On an application of infinitely divisible distributions to quadrature problems

N. TEMIRGALIEV

In [1, 4, 5, 7] Wiener measure was used to estimate one-dimension quadrature formulas. This probabilistic measure corresponds to the Wiener process generated by the normal distribution. The normal distribution is an important, but special case of infinitely divisible distributions.

In the present note we compare the mean quadratic error of the simplest formula, the formula of rectangles with equidistant division, with that of the Monte Carlo method with respect to probabilistic measures, given by infinitely divisible distributions, on classes of functions of one variable. We note that in the multidimensional case analogous problems were considered for Banach measures and some other measures in [6, 2, 9].

Let $F(u)$ be an infinitely divisible distribution with finite dispersion. According to Kolmogorov's theorem ([2], p. 114), $F(u)$ is such if and only if its characteristic function can be written in the form

\[ \varphi(t) = \exp\left\{ \Im t + \int_{-\infty}^{+\infty} \frac{e^{itx} - 1 - itx}{x^2} \, dK(x) \right\}, \]

where $m$ is a real number, $K(x)$ a non-decreasing and bounded function ($K(-\infty) = 0$), and the integrand takes the value $-\frac{t^2}{2}$ at $x=0$. In this case

\[ m = \int_{-\infty}^{+\infty} u \, dF(u) \] and \[ x = K(+\infty) = \int_{-\infty}^{+\infty} u^2 \, dF(u) - m^2. \]

By the same theorem of Kolmogorov we can define for every $0 < \tau \leq 1$ a function $F_{\tau}(u)$ such that

\[ \int_{-\infty}^{+\infty} e^{itu} \, dF_{\tau}(u) = \varphi^{\tau}(t) \equiv \varphi(t) \quad (-\infty < t < +\infty) \]

holds, where $\varphi(t)$ is the characteristic function given by (1).

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As 
\[ \varphi_{\tau+s}(u) = \varphi_{\tau}(u) \cdot \varphi_{s}(u) \quad (0 < \tau \equiv s < \tau+s \equiv 1) \]
we have

\[ F_{\tau+s}(u) = \int_{-\infty}^{+\infty} F_{\tau}(u-v) dF_{s}(v). \]

From (2) and (3) it follows

\[ \int_{-\infty}^{+\infty} u dF_{\tau}(u) = \tau \cdot m, \quad \int_{-\infty}^{+\infty} u^{2} dF_{\tau}(u) = \tau \cdot m + m^{2} \cdot \tau^{2}. \]

For every system of numbers 0 < \tau_{1} < \ldots < \tau_{n} \equiv 1 define the joint distribution as follows:

\[ F_{\tau_{1}, \ldots, \tau_{n}}(\vartheta_{1}, \ldots, \vartheta_{n}) \equiv \int_{-\infty}^{\vartheta_{1}} \cdots \int_{-\infty}^{\vartheta_{n}} dF_{\tau_{1}}(u_{1}) dF_{\tau_{2}-\tau_{1}}(u_{2}-u_{1}) \cdots dF_{\tau_{n}-\tau_{n-1}}(u_{n}-u_{n-1}), \]

and for every permutation \( \sigma \) of \{1, \ldots, n\} set

\[ F_{\tau_{\sigma(1)}, \ldots, \tau_{\sigma(n)}}(\vartheta_{\sigma(1)}, \ldots, \vartheta_{\sigma(n)}) \equiv F_{\tau_{1}, \ldots, \tau_{n}}(\vartheta_{1}, \ldots, \vartheta_{n}). \]

In virtue of (4),

\[ F_{\tau_{1}, \ldots, \tau_{n+1}}(\vartheta_{1}, \ldots, \vartheta_{n}, +\infty, \ldots, +\infty) = F_{\tau_{1}, \ldots, \tau_{n}}(\vartheta_{1}, \ldots, \vartheta_{n}), \]

holds for every \( t_{j} \in (0, 1] \) \((j=1, \ldots, n+k)\), and consequently, by Kolmogorov’s theorem on the existence of stochastic processes, there is a probability space \((\Omega, F, P)\) and a family \(\{x_{\omega}(\tau)\}_{\tau \in [0, 1]}\) of real valued functions \(F\)-measurable with respect to the variable \(\omega \in \Omega\), defined on \(\Omega\) for every \(\tau \in [0, 1]\), and such that \(x_{\omega}(0)=0\) for almost every \(\omega \in \Omega\) and

\[ P\{\omega \in \Omega : x_{\omega}(\tau_{1}) \equiv \vartheta_{1}, \ldots, x_{\omega}(\tau_{n}) \equiv \vartheta_{n}\} = F_{\tau_{1}, \ldots, \tau_{n}}(\vartheta_{1}, \ldots, \vartheta_{n}). \]

holds for every \(0 < \tau_{1} < \ldots < \tau_{n} \equiv 1\).

Moreover, by Kleene’s theorem the orbits \(x_{\omega}(\tau)\) \((0 < \tau \equiv 1)\) are continuous from the right for almost every \(\omega \in \Omega\) and have limits from the left for arbitrary \(0 < \tau \equiv 1\) (see [2], 141—150).

It is easy to check that the process \(\{x_{\omega}(\tau)\}_{\tau \in [0, 1]}\) has independent increments, thus for \(0 < \tau < s \equiv 1\) we have (see also (5)):

\[ \int_{\Omega} x_{\omega}(\tau) \cdot x_{\omega}(s) dP(\omega) = \int_{\Omega} [x_{\omega}(\tau)-x_{\omega}(0)] dP(\omega) \cdot \int_{\Omega} [x_{\omega}(s)-x_{\omega}(\tau)] dP(\omega) \]

\[ = \int_{-\infty}^{+\infty} u dF_{\tau}(u) \cdot \int_{-\infty}^{+\infty} u dF_{s-\tau}(u) + \int_{-\infty}^{+\infty} u^{2} dF_{\tau}(u) = \tau \cdot m + m^{2} \cdot \tau \cdot s. \]