Sets of uniqueness and closed subgroups in Vilenkin groups

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1. Introduction. YONEDA [4] has proved that a closed, measure zero subgroup $H$ of the dyadic group $G$ is a set of uniqueness for the Walsh—Paley system. His proof is highly technical, and it is not clear whether it can be generalized even to the $p$-series case. A key ingredient of his proof is the following result.

**Lemma 1.** There are infinitely many Walsh functions which are identically 1 on $H$.

We give a short proof of this fact which generalizes immediately to any infinite locally compact abelian group. Pontriagin duality (see [1], e.g.) is the main tool.

**Proof.** Let $\hat{G}$ denote the Pontriagin dual. The hypotheses on $H$ imply $G/H$ is an infinite abelian group. Hence, $\text{card}((G/H)^{\sim}) = \infty$, by Pontriagin duality. But $(G/H)^{\sim}$ is isomorphic to $\text{ann}(H) = \{ \psi \in \hat{G} : \psi(H) = 1 \}$, again by Pontriagin duality. Remembering that $\hat{G} = \{ \text{Walsh functions} \}$ we see that the lemma is proved.

This lemma holds for any infinite locally compact abelian group if we substitute “characters” for “Walsh functions”. We will use it to extend Yoneda’s result to a general Vilenkin group (bounded or unbounded). Thus we shall show

**Theorem 1.** If $H$ is a closed, measure zero subgroup of an arbitrary Vilenkin group $G$, then $H$ is a set of uniqueness.

In the Walsh function setting particular examples of such $H$ correspond to Cantor sets $C_\xi$ in $[0, 1]$ with $\xi = 2^{-n}$, $n = 1, 2, \ldots$. Furthermore, our proof deals simultaneously with the Walsh—Paley, Walsh—Kaczmarcz, original Walsh and other orderings.

Our method is different and simpler than Yoneda’s. New ideas here include the application of Pontriagin duality to questions of uniqueness, the consideration of general Vilenkin groups, and rather general ordering of the characters. In section 2 we establish necessary facts and notation for Vilenkin groups. Included is a discussion of orderings for $\hat{G}$. In section 3 we establish a condition sufficient to verify a set $E$ is a set of uniqueness. In section 4 we prove Theorem 1.

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Our knowledge of uniqueness comes from William R. Wade through discussions and through his work which he has generously made available. Particular results crucial to our research are in Wade [3].

2. Vilenkin group prerequisites [2]. A Vilenkin group is defined to be a compact, second countable, zero-dimensional abelian group. It follows there is a chain of subgroups

\[ G = G_0 \supset G_1 \supset \ldots \supset \bigcap_{k=0}^{\infty} G_k = \{0\} \]

forming a neighborhood base at 0. Furthermore, we may assume \( \text{card} (G_{k-1}/G_k) = p_k \) for some primes \( p_k: k = 1, 2, \ldots \). Each \( G_k \) is open and compact. Setting \( N_0 = 1 \) and \( N_k = p_1 p_2 \ldots p_k \) for \( k = 1, 2, 3, \ldots \) gives that \( \text{card} (G/G_k) = N_k \). It follows that a normalized Haar measure \( \mu \) on \( G \) is given by \( \mu(G_k) = N_k^{-1} \) for \( k = 0, 1, 2, \ldots \). The characters \( \psi \) which are identically one on \( G_k \) form a group isomorphic to \( (G/G_k)^* \). It follows there are \( N_k \) such characters. By definition \( \psi \) has conductor \( G_k \), or simply \( k \), if \( \psi \) is identically one on \( G_k \) but not so on \( G_{k-1} \). Thus, there are \( N_k - N_{k-1} \) characters of conductor \( k \) for \( k = 1, 2, \ldots \) and one character of conductor 0. We order the characters by conductor, i.e., we require:

\[ \mathcal{M} \prec m \implies \text{ conductor } \psi_{\mathcal{M}} \leq \text{ conductor } \psi_m. \]

In particular \( \psi_0 \equiv 1 \). Thus, we allow a great deal of arbitrariness in the ordering of the characters. Nevertheless, ordering the characters by conductor guarantees the following well known facts.

(i) Any function \( f \) constant on cosets of \( G_k \) can be written as

\[ f = \sum_{n=0}^{N_k-1} \hat{f}(n) \psi_n \quad \text{where} \quad \hat{f}(n) = \int_G f(x) \overline{\psi_n(x)} \, d\mu(x) \]

is the \( n \)th Fourier coefficient of \( f \). Such functions are called \( G \)-polynomials of level \( k \) or simply polynomials.

(ii) \( \sum_{n=0}^{N_k-1} \hat{f}(n) \psi_n(x) = \frac{1}{|G_k|} \int_{G_k} f \, d\mu \quad \text{for any} \quad f \in L^1. \)

Here \( G_k(x) = x + G_k \). We define \( m \perp n \) for nonnegative integers \( m, n \) by the equation \( \psi_{m+n} = \psi_m \psi_n \). We let \( \perp \) be the operation inverse to \( \perp \).

Let us now emphasize that our sets of uniqueness are with respect to the full set of partial sums.

Definition. A set \( E \subseteq G \) is a set of uniqueness for \( \{\psi_n\}_{n=0}^\infty \) if \( \sum_{n=0}^\infty a_n \psi_n(x) = 0 \) for every \( x \) outside \( E \) implies that the complex numbers \( a_n \) are all zero.