Original articles

Artistic design with fractal matrices

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A simple recursive procedure that grows integer matrices with a fractal nature is presented. Colorful translations of subsets of these matrices produce aesthetically appealing patterns that extend the known classes of fractal images. This is illustrated with several examples.

Key words: Fractals - Computer art - Patterns - Textures - Ornaments - Tilings - Carpets - Dynamical systems

Introduction

Fractal matrices is the name given to the class of unbounded matrices obtained by continued application of the recursion (Barbé 1992)

\[ E_{m+1} = E_1 \oplus E_m. \]  

\[ E_1 \] is an \( r \times s \) (rows \times columns) matrix of integers \( e_1(i,j), 0 \leq i < r, 0 \leq j < s \). It is called the generator matrix. \( A \oplus B \) denotes the so-called bigsum operation between two matrices \( A \) and \( B \): it is the matrix obtained by replacing each element \( a(i,j) \) in \( A \) by the matrix \( B \) to which first \( a(i,j) \) has been added to all its elements. Equation (1) thus defines a two-dimensional symbolic-substitution dynamical system (Mozes 1989), where the symbols are integers and the substitution rule is arithmetical in nature.

The class of matrices obtained through Eq.(1) for different generator matrices is infinite. It emerged from a generalization of the fractal-state space associated with a particular dynamical system known as cellular automaton (Barbé 1990). These matrices also form an extension of a special subset in the class of iterated Kronecker products, as described by Shallit and Stolfi (1989).

From the definition in Eq.(1), it follows that \( E_m \) is an \( r^m \times s^m \) matrix. \( m \) will be designated as the logsize, an abbreviation of “logarithmic size”, of the matrix. Choosing \( e_1(0,0)=0 \) makes \( E_{m+1} \) grow from \( E_m \) by just adding new elements to the right, to the bottom, and to the right bottom of \( E_m \). As an example, consider the generator matrix

\[ E_1 = \begin{bmatrix} 0 & -4 & 1 \\ 1 & 2 & 0 \end{bmatrix}. \]  

It follows that

\[ E_2 = \begin{bmatrix} 0 & -4 & 1 & -4 & -8 & 3 & 1 & 3 & 2 \\ 1 & 2 & 0 & -3 & -2 & -4 & 3 & 1 \\ 1 & -3 & 2 & 2 & -3 & 3 & 0 & -4 & 1 \\ 2 & 3 & 1 & 3 & 4 & 2 & 1 & 2 & 0 \end{bmatrix}. \]  

\[ E_2 \] already shows how Eq.(1) produces an intricate distribution of integers over the growing matrix. It can be shown that

\[ E_m = E_k \oplus E_{m-k} = E_{m-k} \oplus E_m \]  

for all \( k: 0 < k < m \). This means that \( E_m \) can be grown in a way that doubles the logsize at each step according to the sequence \( E_1, E_2, E_4, E_8 \ldots \). Equation (4) also reflects a particular similarity between \( E_m \) and some of its parts in the following sense: when \( E_m \) is partitioned in submatrices of the size of \( E_k \) (the so-called logsize-\( k \) agglomerates of \( E_m \)), the resulting matrix of logsize-\( k \) agglomer-
The similarity referred to concerns a structural equivalence under a 1-1 mapping: i.e., two equally sized matrices $A$ and $B$ whose elements are symbols (possibly of a different nature in $A$ and $B$) are structurally similar when equal/different elements in $A$ correspond to equal/different elements in $B$ under elementwise mapping, and vice versa. Consider e.g. the matrices $E_1$ and $E_2$ above: the matrix, whose elements are the sub-matrices in the partitioning of $E_2$ as shown, is

Figs. 1 and 2. $3^5 \times 3^5 (=243 \times 243)$ $E_5$-matrices generated respectively from the matrices $A$ and $B$

Figs. 3 and 4. $4^4 \times 4^4 (=256 \times 256)$ $E_4$-matrices with the same generator matrix $C$, but with different palettes