Interior regularity operators

R. E. WHITE

Introduction. The study of regularity of solutions of linear differential equations consists of determining the level of smoothness of solutions to $Pu = f$ when $f$ has a given level of smoothness. The operator $P = \frac{\partial^2}{\partial x \partial y}$ does not have this property; the solution of $Pu = 0$ need not have continuous derivatives of order higher than one. However, many important linear differential operators do have in some sense interior regularity. The following are just a few important results about interior regularity of linear differential operators. In 1904 in a paper written by Bernstein [1] it was proved that elliptic operators with analytic coefficients must have analytic solutions. Later in a paper by Petrowsky [11] elliptic operators with constant coefficients were characterized by the fact that the homogeneous equation must have analytic solutions. Interior regularity of solutions up to Gevrey classes of $C^\infty$ which are not analytic has been studied in Hörmander [4] for the semielliptic operators with constant coefficients. Weyl [18] in 1940 showed that weak solutions of the Laplace equation must be in $C^\infty$. This lead to the definition of hypoelliptic operators which were characterized in 1955 by Hörmander [2] for the constant coefficient case and in 1973 by White [19] for the case where the coefficients are in $C^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^n$.

All of the above results are concerned with a degree smoothness in the interior to solutions of partial differential equations. The least smooth types of solutions considered are in $C^\infty(\Omega)$. It is the purpose of this paper to introduce a notion of interior regularity that will measure smoothness of solutions which may not be in $C^\infty$. Since the early 1960's a great deal of the interior regularity research has been done on the larger class of operators called pseudodifferential operators e.g. see Oleinik and Radkevič [9] and Hörmander [3]. Even though this class contains the differential operators with $C^\infty(\Omega)$ coefficients, it seems particularly inept for the study of differential operators with coefficients which are not in $C^\infty(\Omega)$. It

Received February 23, 1979.
is because of the many physical problems with variable coefficients which are not in $C^m(\Omega)$ that this study is undertaken. The interior regularity of the solutions is needed for error estimates of numerical schemes e.g. see Weinberger [17]. Moreover, in the case of nonlinear problems the regularity of the solution to the associated linear problem implies that the solution map is compact e.g. see Kuiper [6] or that a known sequence of iterates converges e.g. see Sattinger [12]. Thus the property of regularity of solutions is fundamental not only to the general theory of partial differential equations but is of essential importance to the application of partial differential equations to physical problems.

Of course the question of interior regularity for specific classes of operators with variable coefficients which are not in $C^m(\Omega)$ has not been neglected. In particular, for elliptic operators see Stampacchia [15], for parabolic operators see Ladyzenskaja, Solonnikov and Uralceva [7] and for pseudoparabolic operators see Showalter and Ting [14]. It is not necessarily the main purpose of this paper to extend specific interior regularity results; although, this is done in Theorem 2. Rather the purpose is to define a notion of interior regularity for differential operators with variable coefficients which are not in $C^m(\Omega)$ and then to study the properties of these operators as a class. This was done by Schwartz [13] (also see Trèves [16]) for the class of hypoelliptic operators. Thus this paper fills a neglected area in the general theory of partial differential equations.

In section one we define interior regularity operators on $\Omega$ and give some examples. One example is formally hypoelliptic operators with coefficients which are not in $C^m$. Another is the second order equations of nonnegative characteristic form with coefficients in $C^m$. Section two contains a characterization of interior regularity operators on $\Omega$. This characterization will be in terms of a priori inequalities similar to those that characterize hypoelliptic operators on $\Omega$. Section three contains some innate properties of interior regularity operators on $\Omega$. These properties are similar to those for hypoelliptic operators on $\Omega$. For example we prove that an interior regularity operator with suitably smooth coefficients will imply that there exists a local fundamental solution to the transpose of the interior regularity operator on $\Omega$.

**Section 1.** We will study interior regularity in the context of the spaces of distributions defined by L. Hörmander.

Let $\hat{u}(\xi)$ be Fourier transform of $u \in S' \equiv$ temperate distributions, $B_{p,k} \equiv \{u \in \mathcal{S}' : \hat{u}(\xi) \text{ is measurable}, \, \hat{u}k \in L_p \text{ and } k \text{ is a temperate weight function}\}$,

$$B_{p,k}^{\text{loc}}(\Omega) \equiv \{u \in \mathcal{D}'(\Omega) : \phi u \in B_{p,k} \text{ for all } \phi \in C^\infty_c(\Omega)\}.$$  

The properties and notation of these spaces may be found in Hörmander [4]. Three