MAXIMAL SUBGROUPS OF THE ORTHOGONAL GROUP

Stephen Pierce *)

Let $V$ be an $n$-dimensional regular quadratic space over a field $K$ of characteristic not 2. Assume $n \geq 4$. Let $W$ be a regular hyperplane and $v$ a nonzero vector orthogonal to $W$. Suppose every regular hyperplane in $W$ is universal. If $\sigma$ is an isometry of $V$ not leaving $W$ invariant, then $\sigma$, together with the isometries of $W$, generate the orthogonal group of $V$, with one exception.

Let $V$ be a vector space over a field $K$ of characteristic not 2. Equip $V$ with a symmetric, regular bilinear form $B(x,y)$, that is $B(x,V) = 0$ only if $x = 0$. We write $B(x,y) = (x,y)$ and set $Q(x) = (x,x)$. Any $\sigma$ in $\text{End} \ V$ satisfying $Q(\sigma x) = Q(x)$ for all $x$ is an isometry of $V$ and the group of all isometries is written $O(V)$. Let $K^* = K\setminus\{0\}$ and write $dV$ for the discriminant of $V$ as a member of $K^*/K^{*2}$. For a subset $S$ of $V$, let $\langle S \rangle$ be the linear span of $S$ and for a subset $S$ of $O(V)$, let $\langle S \rangle$ be the subgroup generated by $S$. A nonzero vector $x$ in $V$ is isotropic if $Q(x) = 0$ and a subspace $W$ of $V$ is isotropic if $W$ has an isotropic vector. If every nonzero vector in $W$ is isotropic, $W$ is totally

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isotropic. The index or Witt index of $V$ is the dimension of a maximal totally isotropic subspace. Put $W^\perp$ for the orthogonal compliment of $W$ and $O(W)$ the isometries of $V$ which fix $W^\perp$ pointwise. If $W$ is regular, we say $W$ is universal if every $\alpha$ in $K^*$ is $Q(w)$ for some $w$ in $W$. If $Q(x) \neq 0$, let $\tau_x$ be the symmetry in $O(V)$ mapping $x$ to $-x$. If $E$ is a basis of $V$ and $\sigma \in O(V)$, $[\sigma]_E$ is the matrix representation of $\sigma$. Use [1] or [5] to reference terminology.

In preparing a recent paper [6] Watkins and the author needed to answer a question about generating $O(V)$. We proved the following result.

**Theorem.** Let $V$ be a regular quadratic space of dimension $n \geq 4$ over a field $K$ of characteristic not 2. Let $W$ be a regular hyperplane in $V$ such that every regular hyperplane in $W$ is isotropic. Pick $\sigma$ in $O(V)$ such that $\sigma W \neq W$. Then $\langle O(W), \sigma \rangle = O(V)$.

It is the purpose of this paper to expand on this idea by relaxing the conditions on $W$. It is well known that $O(V)$ is generated by symmetries [5] and there is plenty of literature on generating questions in classical groups; see, for example, [2], [3], [4]. We particularly cite the work of Wong, [7], [8], who examined generation of classical groups by certain subgroups.

If $W$ is a regular hyperplane in $V$ and $\sigma \in O(V)$ with $\sigma W \neq W$, put $G = \langle O(W), \sigma \rangle$. Our results give some sufficient conditions for $G$ to be $O(V)$, but first we give two examples to show that $G$ can sometimes be a proper subgroup.

**Example A.** Let $V = \mathbb{R}^n$, $n \geq 2$, and let $\{e_1, \ldots, e_n\}$ be the standard basis of $V$. Put $W = \langle e_2, \ldots, e_n \rangle$ and define $Q(x) = -x_2^2 + x_2^2 + \cdots + x_n^2$. Choose $z$ in $V \setminus W$ such that $Q(z) > 0$, and let $\sigma = \tau_z$. Then $G = \langle O(W), \sigma \rangle$ is in the kernel of the spinor norm on $O(V)$, but $V$ is isotropic, so the spinor norm is surjective. Thus $G \neq O(V)$.

Example A suggests that we should at least assume $W$ is