Extension of functions, which are traces of functions belonging to $H^k$ on an arbitrary subset of the line

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Introduction

WHITNEY [13] obtained a necessary and sufficient condition for a function $f=f(x)$ defined on an arbitrarily given closed subset $E$ of $\mathbb{R}$ to be the trace of some function $f \in C^k$ i.e. the trace of a function having a continuous $k$-th derivative on $\mathbb{R}$. This conditions can be stated as follows. For each point $x \in E$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that if $x_0, \ldots, x_k$ and $x'_0, \ldots, x'_k$ are any two sets of distinct points of $E$ contained in the $\delta$-neighbourhood of $x$ then $\| [x_0, \ldots, x_k; f] - [x'_0, \ldots, x'_k; f] \| < \varepsilon$ where $[x_0, \ldots, x_k; f]$ denotes the usual divided difference of the function $f$. See also MERRIEN [10].

In this paper we shall consider the classes $H^k$ of functions $f$ whose $k$-th moduli of continuity do not exceed the majorant function $\varphi = \varphi(t)$.

JONSSON [8] found the respective condition in the case $\varphi(t) = t^{k-1}$ in terms of inequalities for the derivatives of interpolation polynomials. SCHWARZMAN [12] and DJADYK with the author [6] proved that, for the class $H^k_2$, such a condition is that the function $f$ belongs to the Dzjadyk class $H^k_2$ on $E$ [5, p. 176].

The main result of the present article is Theorem 5 (an extension theorem). Together with Theorem 1, it provides a necessary and sufficient condition for a function $f$ defined on an arbitrary set $E \subset \mathbb{R}$ to belong to $H^k$. In particular, for the class $W^k = H^k$ this condition has the form $\| [x_0, \ldots, x_k; f] \| \leq 1$.

1. Throughout the paper we shall make use of the following notations (cf. [5], [7]):

$k$ stands for positive integers,

$$[x_0, x_1, \ldots, x_k, f] = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_k)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_k)} + \ldots + \frac{f(x_k)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})}$$

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is the divided difference of $(k+1)$-th order of the function $f$ associated with the point $x_i \in \mathbb{R}$, $i=0, k$.

\[ L(x,f,x_1,\ldots,x_k) = f(x_1) + \frac{1}{k!} (x-x_1) + \frac{1}{k-1} (x-x_1) (x-x_2) + \ldots + [x_1,\ldots,x_k] (x-x_1) \ldots (x-x_{k-1}), \]

where $[x_1,\ldots,x_k] = [x_1,\ldots,x_j,f]$, denotes the Lagrangean polynomial of at most $(k-1)$-th degree in $x$ interpolating the function $f(x)$ in the points $x_i \in \mathbb{R}$, $i=1, k$. Here we assume that the function $f(x)$ is defined at the points $x_i$, $i=1, k$.

\[ g_k(x,f,a,b) \overset{df}{=} f(x) - L(x,f,a,a+\frac{1}{k-1} (b-a),\ldots,a+\frac{k-2}{k-1} (b-a),b), \]

where $k>1$, $b>a$.

Terms of the form $C_i$ will always denote different constants depending only on $k$, i.e. for fixed $k$, they are absolute constants.

$I$ stands for one of the sets $[a, b]$ or $[a, \infty)$ or $(-\infty, b]$ or $\mathbb{R}$.

\[ \|f\|_I \overset{df}{=} \text{ess sup}_{x \in I} |f(x)|, \]

\[ \Delta_k^b(f,x_0) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x_0+ih) \]

is the $k$-th difference of the function $f$, associated with the points $x_0+ih$, $i=0, k$.

\[ \omega_k(t,f,I) = \sup_{h \in [0,t]} \sup_{x \in I, x+kh \in I} |\Delta_k^b(f,x_0)| \]

denotes the continuity modulus of $k$-th order of the continuous function $f(x)$ on $I$.

$\Phi^k$ is the class of all continuous functions $\varphi(t)$ on $[0, \infty)$ such that $\varphi(0)=0$, $\varphi(t)$ does not decrease on $[0, \infty)$ and $t^{-k}\varphi(t)$ does not increase on $(0, \infty)$. We shall call the functions $\varphi(t) \in \Phi^k$ $k$-majorants.

$MH[k,I,\varphi(t)]$ is the class of all continuous functions $f(x)$ on $I$ satisfying the inequality $\omega_k(t,f,I) \leq M\varphi(t)$, $M=\text{const}$, $\varphi(t) \in \Phi^k$.

$MW^k_I = MH[k,I,\varphi(t)]$ is the class of all functions admitting a locally absolute continuous $(k-1)$-th derivative on $I$ and fulfilling $\|f^{(k)}\|_I \leq M$, $M=\text{const}$.

\[ H[k,I,\varphi(t)] \overset{df}{=} 1 \cdot H[k,I,\varphi(t)], \quad W^k_I = 1 \cdot W^k_I. \]

Remark. Given a continuous function $f(x)$ on $[a, b]$, the formula [1, p. 519] $\varphi(t) = t^k \sup_{u \geq t} u^{-k} \omega_k(u,f,[a,b])$ defines a $k$-majorant function $\varphi(t)$ such that $f$ belongs to $H[k,[a,b],\varphi(t)]$.

\[ \min \{a,0\} = \min \{0,a\} = a \quad \text{for} \quad l > m : \sum_{i=l}^{m} 0 = 0, \quad \prod_{i=l}^{m} 1 = 1, \quad l, m = 0. \]