Extension of functions, which are traces of functions belonging to $H_k^p$ on an arbitrary subset of the line

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Introduction

WHITNEY [13] obtained a necessary and sufficient condition for a function $f=f(x)$ defined on an arbitrarily given closed subset $E$ of $\mathbb{R}$ to be the trace of some function $f \in C^k$ i.e. the trace of a function having a continuous $k$-th derivative on $\mathbb{R}$. This conditions can be stated as follows. For each point $x \in E$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that if $x_0, \ldots, x_k$ and $x'_0, \ldots, x'_k$ are any two sets of distinct points of $E$ contained in the $\delta$-neighbourhood of $x$ then $|[x_0, \ldots, x_k; f] - [x'_0, \ldots, x'_k; f]| < \varepsilon$ where $[x_0, \ldots, x_k; f]$ denotes the usual divided difference of the function $f$. See also MERRIEN [10].

In this paper we shall consider the classes $H_k^p$ of functions $f$ whose $k$-th moduli of continuity do not exceed the majorant function $\varphi = \varphi(t)$.

JONSSON [8] found the respective condition in the case $\varphi(t) = t^{k-1}$ in terms of inequalities for the derivatives of interpolation polynomials. SCHWARZMAN [12] and DZJADYK with the author [6] proved that, for the class $H_2^p$, such a condition is that the function $f$ belongs to the Dzjadyk class $H_2^p$ on $E$ [5, p. 176].

The main result of the present article is Theorem 5 (an extension theorem). Together with Theorem 1, it provides a necessary and sufficient condition for a function $f$ defined on an arbitrary set $E \subseteq \mathbb{R}$ to belong to $H_k^p$. In particular, for the class $W^k = H_k^k$ this condition has the form $|[x_0, \ldots, x_k; f]| \leq 1$.

1. Throughout the paper we shall make use of the following notations (cf. [5], [7]):

$k$ stands for positive integers,

$$[x_0, x_1, \ldots, x_k, f] = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_k)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_k)} + \cdots + \frac{f(x_k)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_k-1)}$$

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is the divided difference of \((k+1)\)-th order of the function \(f\) associated with the point \(x_i \in \mathbb{R},\ i=0, k\).

\[
L(x, f, x_1, \ldots, x_k) = f(x_1) + [x_1, x_2](x-x_1) + [x_1, x_2, x_3](x-x_1)(x-x_2) + \\
+ \ldots + [x_1, \ldots, x_k](x-x_1)\ldots(x-x_{k-1}),
\]

where \([x_1, \ldots, x_j] = [x_1, \ldots, x_j, f]\), denotes the Lagrangean polynomial of at most \((k-1)\)-th degree in \(x\) interpolating the function \(f(x)\) in the points \(x_i \in \mathbb{R},\ i=1, k\). Here we assume that the function \(f(x)\) is defined at the points \(x_i,\ i=1, k\).

\[
g_k(x, f, a, b) \overset{df}{=} f(x) - L\left(x, f, a + \frac{1}{k-1} (b-a), \ldots, a + \frac{k-2}{k-1} (b-a), b\right),
\]

where \(k > 1,\ b > a\).

Terms of the form \(C_i\) will always denote different constants depending only on \(k\), i.e. for fixed \(k\), they are absolute constants.

\(I\) stands for one of the sets \([a, b]\) or \([a, \infty)\) or \((-\infty, b]\) or \(\mathbb{R}\).

\[
\|f\|_I \overset{df}{=} \text{ess sup}_{x \in I} |f(x)|,
\]

\[
A_k^x(f, x_0) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x_0 + ih)
\]
is the \(k\)-th difference of the function \(f\), associated with the points \(x_0 + ih,\ i=0, k\).

\[
\omega_k(t, f, I) = \sup_{h \in [0, t]} \sup_{x \in I, x+kh \in I} |A_k^x(f, x_0)|
\]
denotes the continuity modulus of \(k\)-th order of the continuous function \(f(x)\) on \(I\).

\(\Phi^k\) is the class of all continuous functions \(\varphi(t)\) on \([0, \infty)\) such that \(\varphi(0)=0, \varphi(t)\) does not decrease on \([0, \infty)\) and \(t^{-k}\varphi(t)\) does not increase on \((0, \infty)\). We shall call the functions \(\varphi(t) \in \Phi^k\) \(k\)-majorants.

\(MH[k, I, \varphi(t)]\) is the class of all continuous functions \(f(x)\) on \(I\) satisfying the inequality \(\omega_k(t, f, I) \leq M\varphi(t),\ M=\text{const}\), \(\varphi(t) \in \Phi^k\).

\(MW^k_I = MH[k, I, r^k]\) is the class of all functions admitting a locally absolute continuous \((k-1)\)-th derivative on \(I\) and fulfilling \(\|f^{(0)}\|_I \leq M,\ M=\text{const}\).

\[
H[k, I, \varphi(t)] \overset{df}{=} 1 \cdot H[k, I, \varphi(t)], \quad W^k_I = 1 \cdot W^k_I.
\]

Remark. Given a continuous function \(f(x)\) on \([a, b]\), the formula \([1, \text{p. 519}]\)
\[\varphi(t) = t^k \sup_{u \geq t} u^{-k} \omega_k(u, f, [a, b])\]
defines a \(k\)-majorant function \(\varphi(t)\) such that \(f\) belongs to \(H[k, [a, b], \varphi(t)]\).

\[
\min \{a, 0\} = \min \{0, a\} = a; \quad \text{for} \quad l > m: \quad \sum_{i=l}^m = 0, \quad \prod_{i=l}^m = 1, \quad \overline{l, m} = 0.
\]