SOME NEW INVARIANT PAIRS \((t,3)\) FOR

PROJECTIVE HJELMSLEV PLANES

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Associated with every finite projective Hjelmslev plane is an invariant pair \((t,r)\): \(t\) is the number of neighbours of a given point on a given line passing through it and \(r\) is the order of the underlying projective plane. The Drake-Lenz method \([2],[3]\) of using auxiliary matrices for the constructions of projective Hjelmslev planes has become standard by now. This paper is intended to give some new constructions of projective Hjelmslev planes with invariant pairs \((t,3)\) by making use of the generalization and improvement of the Drake-Lenz theorem\([3]\) obtained by the author in \([6]\) and \([7]\). The results of this paper add 8 new values to the list \(([5], \text{example 3.7(ii)}\) ) of invariant pairs \((t,3)\) with \(t \leq 1,000\) for projective Hjelmslev planes.

1. Prerequisites

For an introduction to projective Hjelmslev planes, \([3]\) or \([4]\) is a good source. All the relevant definitions and results, in particular, the concept of a set of auxiliary matrices and its use in the construction of projective Hjelmslev plane may be found in \([2]\) and \([3]\). We shall be dealing with finite structures only. A projective Hjelmslev plane with an invariant pair \((t,r)\) will be called a PH-plane-\((t,r)\) or a \((t,r)\)-PH-plane. As in \([6]\), we define the following sets:

\[
S_H = \{ (t,r) : \text{there exists a } (t,r)\text{-PH-plane}\},
\]

\[
S_H(3) = \{ t : (t,3) \in S_H \},
\]

and \(S_H(3)\) = \{ \(t : t \in S_H(3)\) and \(t \leq 1,000\)\}.

For positive integer \(n\), \(S(n)\) will denote the set \(\{0,1,\ldots,n-1\}\) of integers from 0 to \(n-1\). All the matrices considere in this paper are \((0,1)\)-square matrices. 0 and J will denote the zero matrix and the matrix with all entries 1, respectively. By the dual of a statement, we shall mean the statement obtained from the original statement by interchanging the words row and column. For a matrix \(A\), \(I(A)\) will denote the set of flags of \(A\), i.e., \(I(A) = \{(i,j) : a_{ij} = 1\}\), where \(A = (a_{ij})\). Script letters will stand for a set of (auxiliary) matrices.
Let $f$ be a map, $f : I(A) \rightarrow S$. The map $f$ will be called a labeling map (on $A$) if (1), (2) and the duals of (1) and (2) are satisfied.

1. Let, for any row $i \in S(n)$, $f_{(i)}$ denote the restriction of the map $f$ to the $i$-th row, i.e.,
$$f_{(i)} : \{(i,j) : a_{ij} = 1, j \in S(n)\} \rightarrow S.$$ Then $f_{(i)}$ is a surjection onto $S$.

2. Given $i,j \in S(n)$, such that the inner product of the $i$-th and $j$-th row is non-zero in $A$, there exist $k,m \in S(n)$, $k \neq m$, such that $a_{ik} = a_{im} = a_{jk} = a_{jm} = 1$ and $f(i,k) \neq f(j,k)$ and $f(i,m) \neq f(j,m)$.

**Theorem 1.2** ([6], theorem 3.2). Let $\mathcal{A} = \{A_0, A_1, \ldots, A_r\}$ be a set of $r+1$ auxiliary matrices of order $t^2$. Let $q$ be an integer satisfying
$$2(r+1) \leq q+1 \leq t(r+1).$$

Further, let there exist a set of $q+1$ auxiliary matrices of order $s^2$. Let $q+1 = \sum_{\alpha=0}^{q} q_{\alpha}$, $2 \leq q_{\alpha} \leq t$, for each $\alpha \in S(r+1)$. If, for every $\alpha \in S(r+1)$, there exists a labelling map $f_{\alpha} : I(A_{\alpha}) \rightarrow S(q_{\alpha})$, then there exists a set of $r+1$ auxiliary matrices of order $t^2s^2$ and consequently a $(ts,r)$-PH-plane.

**Construction 1.3.** In [6] and [7], the constructions depend on the use of special kinds of auxiliary matrices called the partition matrices. These matrices are constructed as follows. Let an affine plane of order $r$ exist. Number the points of this affine plane by the elements of $S(r^2)$ and the parallel classes of lines by the elements of $S(r+1)$. Define the matrices $D_{\alpha}$, $\alpha \in S(r+1)$, by
$$D_{\alpha} = (d_{ab}^{\alpha}), a,b \in S(r^2),$$
$$d_{ab}^{\alpha} = 1, \text{ if the points } a \text{ and } b \text{ are joined by a line in the } \alpha\text{-th parallel class},$$
$$= 0, \text{ otherwise.}$$