On the strong approximation by the \((C,\varpi)\)-means of Fourier series. II

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1. In the article [3] we have proved several theorems about the approximation order of certain strong means. We continue the considerations in this paper.

To avoid unnecessary repetitions we keep the notations of [3] and use them without any more reference (see for them especially the end of Section 1 in [3]).

Beyond the results mentioned in [3] there is another theorem concerning with this kind of approximation.

Theorem A [1, Theorem 4]. Let us suppose that \(f \in \text{Lip} \, \delta, \, 0 < \delta < 1, \, \alpha > -1/2, \, p > 0 \) and \( p \alpha > -1 \). If \( \lambda = \{ \lambda_n \} \) is a non-decreasing sequence of integers for which \( \lambda_1 = 1, \, \lambda_{n+1} - \lambda_n = 1, \) then

\[
V_n(f, \alpha, \lambda, p; x) = \left\{ \frac{1}{\lambda_n} \sum_{k=n+1}^{\infty} |\sigma_k^\alpha(x) - f(x)|^p \right\}^{1/p} =
\begin{cases}
O \left( n^{-\delta} \left( \frac{n}{\lambda_n} \right)^{1/p} \right) & \text{if } p\delta < 1, \\
O \left( \lambda_n^{-\delta} \left( 1 + \log \frac{n}{n-\lambda_n+1} \right)^{1/p} \right) & \text{if } p\delta = 1, \\
O \left( \lambda_n^{-1/p} (n-\lambda_n+1)^{1/p-\delta} \right) & \text{if } p\delta > 1.
\end{cases}
\]

Now we generalize this result. Theorem 1 will show that (1.1) does not give the best estimation.

Let

\[
\tilde{V}_n(f, \alpha, \lambda, p; x) = \left\{ \frac{1}{\lambda_n} \sum_{k=n+1}^{\infty} |\sigma_k^\alpha(x) - f(x)|^p \right\}^{1/p}.
\]

Theorem 1. Let \( 0 < \alpha < -1/2, \, p > 0, \, p\alpha > -1 \) and \( f \in H^\infty \).

(i) If \( \lambda_n = O(1) \) or \( \omega(\delta) = O(\delta) \), then

\[
V_n(f, \alpha, \lambda, p; x) = O(\tau_n(\alpha, \lambda, p, \omega))
\]
and
\begin{equation}
\tilde{V}_n(f, \alpha, \lambda, p; x) = O(\tau^*_n(\alpha, \lambda, p, \omega)) ,
\end{equation}
where
\begin{equation}
\tau_n(\alpha, \lambda, p, \omega) = \left\{ \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} \left( \omega^*(\frac{1}{k}) \right)^p \right\}^{1/p} + \left( \frac{\lambda_n}{n} \right)^{\omega} \left( \frac{1}{n} \right)
\end{equation}
and
\begin{equation}
\tau^*_n(\alpha, \lambda, p, \omega) = \left\{ \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} \left( \omega^{**}(\frac{1}{k}) \right)^p \right\}^{1/p} + \left( \frac{\lambda_n}{n} \right)^{\omega} \left( \frac{1}{n} \right).\]

Furthermore, there are functions \( f_1 \) and \( f_2 \) such that \( A, f_2 \in H^\omega \),
\begin{equation}
\limsup_{n \to \infty} \frac{V_n(f_1, \alpha, \lambda, p; 0)}{\tau_n(\alpha, \lambda, p, \omega)} > 0
\end{equation}
and
\begin{equation}
\limsup_{n \to \infty} \frac{\tilde{V}_n(f_2, \alpha, \lambda, p; 0)}{\tau^*_n(\alpha, \lambda, p, \omega)} > 0.
\end{equation}

(ii) If, however, \( \lambda_n = O(1) \) and \( \omega(\delta) = O(\delta) \), that is \( f \in \text{Lip} 1 \), then
\begin{equation}
V_n(f, \alpha, \lambda, p; x) = O\left( \frac{1}{n^{1+\alpha}} \right)
\end{equation}
and
\begin{equation}
V^*_n(f, \alpha, \lambda, p; x) = O\left( \frac{1}{n^{1+\alpha}} \right)
\end{equation}
uniformly in \( x \).

Furthermore, to an arbitrary sequence \( \{\varrho_n\} (\varrho_n > 0) \) tending to zero, we can find \( f_1, f_2 \in \text{Lip} 1 \) for which
\begin{equation}
\limsup_{n \to \infty} V_n(f_1, \alpha, \lambda, p; 0) \frac{n^{1+\alpha}}{\varrho_n} > 0
\end{equation}
and
\begin{equation}
\limsup_{n \to \infty} \tilde{V}_n(f_2, \alpha, \lambda, p; 0) \frac{n^{1+\alpha}}{\varrho_n} > 0.
\end{equation}

If, in particular, \( \omega(\delta) = \delta^\gamma (0 < \gamma < 1) \), then \( \omega^*(\delta) = \delta^\gamma \) and, as \( 1/p > -\alpha \), Theorem 1 gives a sharper estimation than Theorem A.

In the case \( \alpha < 0 \), the results of [3] as well as Theorem 1 all assumed \( 0 > \alpha > -1/2 \) and \( p\alpha > -1 \). The next two theorems show that these assumptions were very essential.

As the means
\begin{equation}
V^*_n(f, \alpha, p; x) = \left\{ \frac{1}{n} \sum_{k=n+1}^{n+1} |\sigma_k(x) - f(x)|^p \right\}^{1/p}
\end{equation}
and
\begin{equation}
\tilde{V}^*_n(f, \alpha, p; x) = V^*_n(f, \alpha, p; x)
\end{equation}