MINKOWSKIAN GEOMETRY, CONVEXITY CONDITIONS AND THE PARALLEL AXIOM

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The problem of characterising Minkowskian spaces is an important problem of that branch of differential geometry in which spaces more general than the complete Riemann and Finsler spaces are studied axiomatically using synthetic geometric methods. The fundamental theorem in this field is the result that a Desarguesian straight G-space in which the parallel axiom holds and the spheres are convex is Minkowskian. However the question as to whether the hypothesis of the space being Desarguesian is necessary or not has remained unsolved for over forty years. It is therefore natural to investigate conditions stronger than the mere convexity of spheres. In this paper such geometric conditions derived from functions which measure the distance between lines and points on lines are studied. Besides characterising the Minkowskian spaces these investigations also bring out the interplay between the parallel axiom and the convexity and linearity conditions.

1. INTRODUCTION. This paper deals with the relations between Minkowskian Geometry (M), various convexity and linearity conditions (A, B, C and D) and the Parallel Axiom (P) in straight spaces. A straight space is a metric space satisfying the Bolzano-Weierstrass Theorem in which any two distinct points \(a, b\) lie on exactly one curve, called the line \(L(a, b)\), isometric to the real axis.  

In a Desarguesian straight space, i.e., a space defined in a convex subset of the real \(n\)-dimensional affine space \(A^n\) whose lines fall on the affine lines (but need not coincide with them) the convexity of the spheres (C) and P guarantee M, see [1, (24.1), p. 144]. (When the meaning is clear from context, we omit the parentheses around

1) We repeat some standard definitions of G-space theory in the introduction to make it widely understandable; in the text and for results we refer to the book [1].
the names of the conditions.)

Whether in general straight spaces \( C \cup P \sim M \) is an open question whose answer has eluded us since over forty years and still does. However, various conditions, some only slightly stronger, characterising \( M \) are known. We mention first the linearity requirement:

A triangle \( abc \) in a straight space and the midpoints \( b' \) of \( a, b \) and \( c' \) of \( a, c \) satisfy \( 2b'c' = bc \).

That this makes the metric Minkowskian is proved in \([1, (36.2), (39.12)]\). Here we show first that a similar hypothesis (A) which is often easier to verify does the same (i.e. \( A \sim M \)).

(A): If \( L \) is a line, \( p \in L \), \( a \not\in L \) and \( a' \) is the midpoint of \( p \) and \( a \) then \( 2a'L = aL \).

We then turn to convexity conditions. For \( n = 2 \), it is known, see \([1, (25.6)]\), that \( B_\mu \cup P \sim M \) where

(B\(_\mu\)): A fixed \( \mu > 1 \) exists such that whenever \( a \neq b \) and \( m \) is the midpoint of \( a, b \) then for \( p \not\in L(a, b) \)

\[ 2pm^\mu - pa^\mu - pb^\mu \leq 0. \]

(B\(_\mu\)) becomes with increasing \( \mu \) (in straight spaces) only seemingly weaker, but leads for \( \mu \to \infty \) to the actually weaker C.

If instead of \( \mu \) we let \( pm \) tend to \( \infty \) when \( p \) traverses a ray (not on \( L(a, b) \)) with origin \( m \) we obtain

(B): \[ \lim_{pm \to \infty} (2pm - pa - pb) = 0. \]