AN EXTENSION OF $A$-STABILITY TO ALTERNATING DIRECTION IMPLICIT METHODS*

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Abstract.
Completely implicit, noniterative, finite-difference schemes have recently been developed by several authors for nonlinear, multidimensional systems of hyperbolic and mixed hyperbolic-parabolic partial differential equations. The method of Douglas and Gunn or the method of approximate factorization can be used to reduce the computational problem to a sequence of one-dimensional or alternating direction implicit (ADI) steps. Since the eigenvalues of partial differential equations (for example, the equations of compressible fluid dynamics) are often widely distributed with large imaginary parts, $A$-stable integration formulas provide ideal time-differencing approximations. In this paper it is shown that if an $A$-stable linear multistep method is used to integrate a model two-dimensional hyperbolic-parabolic partial differential equation, then one can always construct an ADI scheme by the method of approximate factorization which is also $A$-stable, i.e., unconditionally stable. A more restrictive result is given for three spatial dimensions. Since necessary and sufficient conditions for $A$-stability can easily be determined by using the theory of positive real functions, the stability analysis of the factored partial difference equations is reduced to a simple algebraic test.

1. Introduction.

Alternating direction implicit (ADI) methods for parabolic equations were originated by Douglas [10] and Peaceman and Rachford [24]. A general procedure for constructing ADI schemes for multidimensional parabolic equations and the second-order wave equation was devised by Douglas and Gunn [12]. An ADI method for first-order linear hyperbolic systems in two space dimensions was constructed by Gourlay and Mitchell [15].

Recently, completely implicit, noniterative, finite-difference schemes have been developed by several authors for nonlinear, multidimensional systems of hyperbolic [1,25] and mixed hyperbolic-parabolic [2,5,6,19,25] partial differential equations. Lindemuth and Killeen formulated their ADI scheme by following the Douglas, Peaceman-Rachford procedure, Briley and McDonald devised their ADI algorithm by a formal application of the Douglas-Gunn procedure, while Beam and Warming constructed an ADI method by using approximate factorization.

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The linear stability analysis for the algorithms applied to systems of hyperbolic-parabolic equations is in a very rudimentary state. The primary reason is that the operators involved do not commute. In addition, the stability analysis of schemes for mixed hyperbolic-parabolic equations is generally more difficult than the analysis for either type treated separately. This is particularly true for schemes using more than two time levels. In fact, we are aware of only one stability proof for a multistep ADI scheme applied to a model equation with both convective (hyperbolic) and diffusive (parabolic) terms [7].

The eigenvalues associated with a mixed hyperbolic-parabolic system (for example, the equations of compressible fluid dynamics) are often widely distributed with large imaginary parts. Since the eigenvalue spectrum cannot be bounded away from the imaginary axis, A-stable linear multistep integration formulas provide ideal time-differencing approximations. (For the definition of A-stable methods, see, e.g., [8, 14] or section 2.) Although the temporal accuracy of an A-stable linear multistep method (LMM) cannot exceed two [8], this is compatible with the accuracy achievable with typical ADI schemes.

The purpose of this paper is to show that if one uses an A-stable LMM to integrate an evolutionary partial differential equation (PDE) of the form

\[ \frac{\partial u}{\partial t} = (L_x + L_y)u, \]

where \( L_x \) and \( L_y \) are linear scalar differential operators, then one can always construct an ADI scheme by the method of approximate factorization which is also A-stable, i.e., unconditionally stable. Since necessary and sufficient conditions for the A-stability of an LMM can easily be determined by applying the theory of positive real functions [9], the stability analysis of the factored partial difference equations is reduced to a simple algebraic test. Our most general result is for two spatial dimensions with a more restrictive result for three spatial dimensions. We should add that the stability analysis is for simple linear test equations and we have not dealt with noncommuting operators.

In section 2 we briefly review the theory of linear multistep methods. In section 3 we describe a method of constructing an ADI method by starting with a linear multistep method and then using the method of approximate factorization. A linear (model) test equation for partial differential equations is defined (section 4) and then used to analyze the stability of approximate factorization schemes (section 5). The natural extension of approximate factorization methods to three spatial dimensions is discussed in section 6. In section 7 we examine in detail the family of A-stable linear two-step methods. To illustrate the notions of this paper, we write out an ADI method for the three-dimensional heat equation (section 8). Section 9 contains an illustration of a reduced stability range for an approximate factorization formulation which does not follow the formulation of section 3. The connection between this paper and the classic paper on ADI methods by Douglas and Gunn [12] is discussed in section 10. The final section includes a summary of the approximate factorization approach described in this paper.