AN ANALYSIS OF THE EFFECT OF ROUNDED ERRORS ON THE FLOW OF CONTROL IN NUMERICAL PROCESSES

ESKO UKKONEN

Abstract.

Rounding errors may change the flow of control in numerical computing processes by leading to changes in some branching decisions of the process. In this paper some general topological and measure theoretical results associated with this effect of rounding errors are derived. The approach is based on a model of numerical computation related to program schemes. Each computing process specified by the model computes a partial function $R^n \to R^n$ using rational operations and simple tests on real numbers. The topological structure of input point sets in $R^n$ on which the computation follows the same execution path is studied. We also investigate input points, called sensitive, on which rounding errors may change the execution path followed. Conditions concerning computing processes are given which guarantee that the Lebesgue measure of sensitive points approaches zero (i.e. the probability of a branching error gets arbitrarily small) as the precision of the arithmetic increases. Most numerical processes used in practice are easily seen to satisfy these conditions.

1. Introduction.

Consider a computer program containing a test, say "$t > 0$", where the value of $t$ is implemented as a floating-point number. Suppose that the program is executed in finite precision arithmetic. When the test "$t > 0$" is encountered, the cumulative rounding error of $t$ may be so big that the truth value of the test differs from the correct value obtained with infinite precision. Since a wrong truth value may lead to a wrong branching decision, the arithmetic operations performed may also change. In this way rounding errors may make the flow of control unstable.

The instability of the flow of control described above has been given little consideration in the analysis of numerical computation. In the wide literature on numerical stability we only mention [2] and [9] as typical examples of general analyses in which interest is turned to straight-line programs without branching decisions. On the other hand, the uncertainty of tests is discussed as a characteristic feature of numerical computation by some authors [5], [11], [16]. A more extensive analysis of the flow of control in computing processes involving rounding errors is presented in [14].

A reason for investigating the effect of rounding errors on the flow of control is that the effect decreases the reliability of the automatic techniques for rounding error analysis. These well-known techniques in which the computer is asked to produce both an answer and an indication of its accuracy are described e.g. in [13, Ch. 7], [8], [10]. If rounding errors change the sequence of arithmetic operations, the indication of the accuracy often merely states how good the answer is when compared with the answer obtained if the same incorrect operation sequence is performed in infinite precision. Thus automatic methods are generally valid only when errors do not change the operation sequence.

In this paper, which is related to the work by Tienari [14] and also inspired by Knuth [6], we derive in a rather general setting some topological and measure theoretical results associated with rounding errors and the flow of control. Our approach is based on a model of numerical computation, i.e., on a formalism to define numerical processes. A numerical process defined by the model computes a partial function from the input space $\mathbb{R}^n$ (the Euclidean $n$-space) into the output space $\mathbb{R}^m$ for some $n$ and $m$ using the usual rational operations on real numbers, input and output operations, and simple tests of the form "$t > 0$". In fact, the model is a special case of the tree schemes occurring in the theory of program schemes; see e.g. [4, pp. 3–10—3–18]. Rounding errors are modelled with relative perturbations.

After introducing in Chapter 2 our model of numerical computation we consider in Chapter 3 the flow of control from the topological point of view. The input point set in $\mathbb{R}^n$ in which the process follows the same execution path for a finite (bounded) number of steps is shown to have a finite (bounded) number of topological components.

Chapter 4 is devoted to a measure theoretical analysis of input points, called sensitive, on which the rounding errors may change the execution path followed. As a main result conditions concerning numerical processes are given which guarantee that the Lebesgue measure of sensitive points approaches zero as the precision of the arithmetic increases. In other words, if the process satisfies our conditions, the probability that the rounding errors lead to an incorrect execution path can be made arbitrarily small by increasing the precision. This is an important result, since most numerical processes used in practice satisfy these conditions. The Jordan content of sensitive points is also considered.

Chapter 5 gives a few concluding remarks. We point out that the propagation of rounding errors can be investigated in the same way as stability of flow of control is analyzed.

The details not given in this paper can be found in [15].

2. A model of numerical computation.

In this chapter we define a simple model for numerical computing processes which properly represents rounding errors and their effect on the flow of control.