ON A METHOD OF NOBLE FOR SECOND KIND VOLTERRA INTEGRAL EQUATIONS

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Abstract.

We modify Noble's method by replacing the unknown solution in each interval by repeated quadratic and cubic interpolating polynomials and obtain fourth order convergence with precision three and improved accuracy. We also describe two interesting fourth order self-starting methods based on the above concept which show the superiority of the modified method with respect to precision and accuracy. Numerical examples are given to demonstrate the properties of the new methods.

1. Introduction.

Consider a nonlinear Volterra integral equation of the second kind

\[ y(x) = f(x) + \int_a^x K(x,t,y(t)) \, dt, \quad a \leq x \leq b, \]

and assume that \( K \) and \( f \) satisfy all conditions sufficient for (1) to have a unique solution on \([a, b]\).

In the past, methods for approximating the unknown solution of (1) have frequently been obtained by replacing the integral in each interval by a numerical quadrature formula treating \( K(x,t,y(t)) \) as a single function. However, a better method for approximating the unknown solution \( y(x) \) may be obtained by treating \( y(x) \) separately in each interval. Methods obtained for (1) using the above concept will have higher precision and improved accuracy.

In support of the above statement, we consider the following class of nonlinear second kind Volterra integral equations:

\[ y(x) = f(x) + \int_a^x k(x,t)u(y(t)) \, dt, \quad a \leq x \leq b. \]

If \( k(x,t) \) oscillates more rapidly or is less smooth than the solution \( y(x) \), then better approximations for the unknown solution \( y(x) \) can be obtained by treating \( y(x) \) separately.

In [1] a method is described for the numerical solution of (1) in which the above concept has been used to remove some difficulties. However, this method is based on extrapolations.

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Going into the details we reach the conclusion that any method for (1) which has been obtained by treating $K(x, t, y(t))$ as a single function can be easily modified to give better approximations for the unknown solution $y(x)$ by applying the same approximations on $y(x)$ separately. As such modifications also preserve the strong convergence and stability properties of the methods they are most desirable.

The present modified method is simple, economic, stable (stability follows from [2]) and it appears that it has maximum precision and best accuracy in the class of fourth order methods for (1), with one draw-back that it requires one starting value other than $y(a)$. For this reason we also describe two fourth order self-starting methods based on the above concept in Section 4. The first method is explicit and exact only for constants. The second method is implicit and has precision one. Since the accuracy of the modified method depends upon the accuracy of the starting value, we suggest to use these methods for determining the starting value of the modified method with stepsize $h^2/(b-a)$, where $h$ is the stepsize of the modified method. The proof of fourth order convergence of these methods is similar to that for the modified method (described in Section 3) and is therefore omitted.

2. The modified method.

With a fixed stepsize $h$ we define $x_{n+i/j} = a + nh + (i/j)h$, where $n$, $i$ and $j$ are non-negative integers with $j 
eq 0$. We use the notations $y(x_{n+i/j}) = y_{n+i/j}$, $f_{n+i/j}$ and $K_{n+i/j}$ being defined analogously.

We shall also use the following notations throughout this paper:

\begin{align}
(3a) \quad p_n(g; r) &= \frac{1}{2}r(r-1)(r-2)g_{n-1} + (1/2)r(r-1)g_{n+2} , \\
(3b) \quad q_n(g; r) &= -\frac{1}{6}(r-1)(r-2)(r-3)g_{n-1} + (1/6)r(r-2)(r-3)g_{n+3} \\
&\quad + (-1/2)r(r-1)(r-3)g_{n+2} + (1/6)r(r-1)(r-2)g_{n+3} .
\end{align}

Let $N$ be a positive integer and $h = (b-a)/(2N)$. We derive our method by approximating the integral in the following discrete forms of (1). For $n=1(1)N$,

\begin{align}
(4) \quad y_{2n} &= f_{2n} + h \sum_{k=0}^{n-1} \int_0^2 K_{2n}(x_{2k} + rh, y(x_{2k} + rh)) \, dr , \\
\text{and (for } 2n+1 < 2N) \quad (5) \quad y_{2n+1} &= f_{2n+1} + h \sum_{k=0}^{n-2} \int_0^2 K_{2n+1}(x_{2k} + rh, y(x_{2k} + rh)) \, dr \\
&\quad + h \int_0^3 K_{2n+1}(x_{2n-2} + rh, y(x_{2n-2} + rh)) \, dr .
\end{align}

Using the notations of (3), we replace $y(x_{2k} + rh)$ on $[0, 2]$ and $y(x_{2n-2} + rh)$ on $[0, 3]$ by $p_{2k}(y; r) + R_{2k}(r, h)$ and $q_{2n-2}(y; r) + R_{2n-2}(r, h)$ respectively, where