FINDING MINIMAL NESTED POLYGONS

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Abstract.

An algorithm for finding a polygon with minimum number of edges nested in two simple n-sided polygons is presented. The algorithm solves the problem in at most $O(n \log n)$ time, and improves the time complexity of two previous $O(n^2)$ algorithms.

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1. Introduction.

Let one simple polygon lie entirely inside another simple polygon. A third polygon separating the boundaries of these two polygons is called a nested polygon. If the number of edges in the nested polygon is minimal over all the nested polygons, then it is called a minimal nested polygon (which is simply denoted by MNP). Aggarwal et al. [1] first investigated the problem of finding an MNP for two n-sided convex polygons. They provided an $O(n \log n)$ algorithm to solve this problem. Suri and O’Rourke [5] derived an $O(n^2)$ algorithm to solve the problem for two simple n-sided polygons, and Dasgupta and Veni Madhaven [3] presented an $O(n \log n + e)$ algorithm for simple rectilinear polygons, where $e \leq n^2$. The applications of this problem can be found in [1, 3, 7] and in the papers they cited.

In this paper, we classify the problem into two cases by identifying the fixing ports of the two polygons (Section 2). We briefly describe an optimal $O(n)$ algorithm to solve the problem with fixing ports (Section 3) and give an $O(n \log n)$ algorithm to solve this problem without fixing ports (Section 4). Our algorithm (combining the two) improves the results in [3, 5]. Throughout this paper, we use $P$ and $Q$ to denote the boundaries of the two given polygons, and adapt the assumptions in [1, 3, 5, 6] that $P$ contains $Q$, $P$ and $Q$ do not share any point, and each of $P$ and $Q$ has $n$ vertices.

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2 The $O(n \log n)$ algorithm was briefly presented in the conference paper [7].

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2. Preliminaries.

DEFINITION. Let \( R \) denote the region containing \( P \) and \( Q \). A spanning segment is a line segment extended maximally in \( R \). A fixing-port of \( P \) and \( Q \) is a sequence of four points \((u \in P, v \in Q, u' \in P, v' \in Q)\) or \((v \in Q, u \in P, v' \in Q, u' \in P)\) on a spanning segment, where \( u \) and \( v' \) are vertices. (Refer to Fig. 1.)

**Fig. 1.** Spanning segments and the fixing ports.

**Lemma 2.1.** \( P \) and \( Q \) contain a fixing port if \( CH(P) \cap Q \neq \emptyset \) or \( CH(Q) \cap P \neq \emptyset \), where \( CH(P) \) (respectively \( CH(Q) \)) is the boundary of the convex hull of \( P \) (respectively \( Q \)).

**Proof.** "\( \rightarrow \)" If \( CH(Q) \cap P \neq \emptyset \), then there is either a vertex \( u \in P \) (or an edge \( uu' \in P \)) which lies on an edge \( vv' \in CG(Q) \) such that \( P \) does not cross \( vv' \), or a piece of \( P \), say \( U' = (x, u, ..., u', y) \), which crosses \( vv' \) and lies inside \( CH(Q) \). (A line \( l \) crosses \( P \), if both sides of \( P \) contain some points of \( l \).) In the former case, \((v, u, v', u')\) is a fixing port by definition, where \( u' \) is the intersection point of the extending \( v v' \) and \( P \). In the latter case, there must exist a spanning segment which is supported by a vertex, say \( u'' \) from \((x, u, ..., u', y)\), and a vertex, say \( v'' \) from \((v, ..., v')\) of \( Q \), and which ends at points \( w \in P \) and \( z \in Q \). Hence, the sequence \((z, u'', v'', w)\) is a fixing port by definition. The proof is similar for \( CH(P) \cap Q \neq \emptyset \).

"\( \leftarrow \)" It follows from the fact that the vertices of a fixing port are either on or inside \( CH(P) \) or \( CH(Q) \).

Suri showed that, given a triangulated polygon with \( n \) sides, a minimum link path (that is, the path with minimum edges) of any two points inside the polygon can be found in \( O(n) \) time [5, Th. 1, p. 108].

It can be proved that a minimum link path, denoted by \( G \), has greedy property: Let \( e, e', e'' \) be three consecutive edges in \( G \), and let \( l, l', l'' \) be their spanning segments. Then, any line segment in \( R \) intersecting \( l \) cannot "stretch" longer than \( l' \) to intersect \( l'' \) at a further point along \( G \).