Part II

NUMERICAL MATHEMATICS
COMPONENT-WISE PERTURBATION ANALYSIS
AND ERROR BOUNDS FOR LINEAR LEAST
SQUARES SOLUTIONS

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Abstract.

Perturbation bounds for the linear least squares problem \( \min_x \|Ax - b\|_2 \) corresponding to component-wise perturbations in the data are derived. These bounds can be computed using a method of Hager and are often much better than the bounds derived from the standard perturbation analysis. In particular this is true for problems where the rows of \( A \) are of widely different magnitudes. Generalizing a result by Oettli and Prager, we can use the bounds to compute a posteriori error bounds for computed least squares solutions.

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1. Introduction.

Consider the linear least squares problem

\[
(1.1) \quad \min_x \|Ax - b\|_2, \quad A \in \mathbb{R}^{m \times n}, \quad m \geq n,
\]

where, in order to avoid discussing subtle questions related to rank deficient problems, we assume that \( \text{rank}(A) = n \).

Let \( \bar{x} \) be an approximate solution to (1.1), and assume that \( \delta A \) and \( \delta b \) are perturbations such that \( \bar{x} \) is the exact solution to the perturbed problem

\[
(1.2) \quad \min_x \|(A + \delta A)\bar{x} - (b + \delta b)\|_2,
\]

and

\[
(1.3) \quad \|\delta A\|_2 \leq \varepsilon_A \|A\|_2, \quad \|\delta b\|_2 \leq \varepsilon_b \|b\|_2.
\]

Then the norm of the error \( \delta x = \bar{x} - x \) can be bounded using standard perturbation results. Let \( A^\dagger \) denote the pseudoinverse of \( A \) and \( \kappa_2 = \|A^\dagger\|_2 \|A\|_2 \) the condi-