Abstract.

The interpolation problem at uniform mesh points of a quadratic spline
\[ s(x_i) = f_i, \quad i = 0, 1, \ldots, N \]
and
\[ s'(x_0) = f'_0 \]
is considered. It is known that \[ \| s - f \|_\infty = O(h^3) \] and \[ \| s' - f' \|_\infty = O(h^2) \], where \( h \) is the step size, and that these orders cannot be improved. Contrary to recently published results we prove that superconvergence cannot occur for any particular point independent of \( f \) other than mesh points where \( s = f \) by assumption. Best error bounds for some compound formulae approximating \( f'_i \) and \( f'^{(3)}_i \) are also derived.

AMS Subject classification: 65D05, 65D07.

Key words: Interpolation, Quadratic, Spline.

1. Introduction.

In this paper the interpolation problem at uniform mesh points of a quadratic spline
\[ s(x_i) = f_i, \quad i = 0, 1, \ldots, N \]
and
\[ s'(x_0) = f'_0 \]
is considered. If \( h \) is the step size then it is known that \( \| s - f \|_\infty \) is of order \( h^3 \) and \( \| s' - f' \|_\infty \) is of order \( h^2 \). These orders are known to be best possible. We will prove that we cannot have superconvergence for any particular point in the domain (except of course for the mesh points where we have \( s = f \)). Indeed, points for which higher orders are required either do not exist or, when they do exist, they change with \( f \). Thus the obtained results contradict those of R. A. Usmani [4] since for certain selected points he obtained higher orders. Also, best error bounds for certain combination formulae based on the values of the derivative of this spline at three consecutive points approximating \( f'_i \) and \( f'^{(3)}_i \) are established and are in contradiction with those of [4].

Received April 1989. Revised January 1990.
2. The interpolation problem.

Let \( \{x_i, \ i = 0,1, \ldots, N\} \) be a uniform partition of \([0,1]\). Denote by \( S_{N,2}^{(1)} \) the linear space of quadratic splines \( s(x) \) verifying
\[
\begin{align*}
(s(x)) & \in C^1[0,1] \\
(s(x)) & \text{ is a quadratic polynomial in each subinterval } [x_i, x_{i+1}].
\end{align*}
\]
Set \( h = x_{i+1} - x_i \), and denote by \( g_i = g(x_i), \ i = 0,1, \ldots, N \), where \( g \) is any real valued function defined in \([0,1]\). First we recall the following well known result.

**Theorem 1.** Given the real number \( f_i, \ i = 0,1, \ldots, N \) and \( f'_0 \), there exists a unique \( s \in S_{N,2}^{(1)} \) such that
\[
\begin{align*}
(s)_i & = f_i \\
(s)'_0 & = f'_0.
\end{align*}
\]

The quadratic spline which satisfies (2.1) in \([x_i, x_{i+1}]\) is
\[
s(x) = f_i A(t) + h s'_i B(t) + f_{i+1} C(t)
\]
where
\[
A = 1 - t^2, \quad B = t - t^2, \quad C = t^2
\]
and \( t = (x - x_i)/h \). The coefficients \( s'_i \) satisfy
\[
\begin{align*}
(s)'_0 & = f'_0 \\
(s')_{i-1} + s'_i & = \frac{2}{h} (-f'_{i-1} + f_i).
\end{align*}
\]

Writing the last equation for \( i \) and \( i + 1 \) and subtracting we get the following alternative formula for computing \( s'_i \):
\[
\begin{align*}
(s)'_0 & = f'_0 \\
(s)'_1 & = -s'_0 + \frac{2}{h} (-f'_0 + f'_1) \\
-s'_i - s'_{i+1} & = \frac{2}{h} (f_{i-1} - 2f_i + f_{i+1}).
\end{align*}
\]

From the computational point of view this formula is slightly better than (2.4) since its condition number is about half that of (2.4).

3. Error bounds.

In this section \( L_\infty \) error bounds are presented for the above quadratic spline and its first derivative in \([0,1]\). \( \|\cdot\| \) is the \( L_\infty \) norm and \( e(x) = s(x) - f(x) \) for \( x \in [0,1] \).