STABILITY OF A RUNGE-KUTTA METHOD FOR THE NAVIER-STOKES EQUATION

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Abstract.

We investigate the linear stability properties of a Runge-Kutta method for the Navier-Stokes equations. The theoretical stability limit is compared with that encountered in numerical simulations of an initial-boundary value problem. Numerical results from simulation in 3D are presented.

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1. Introduction.

The linear stability of a Runge-Kutta method for solving the compressible Navier-Stokes equations is studied in this paper. The study is based on Fourier transformation of the linearized spatial operator. The aim of this paper is to consider the full, unsplit spatial operator resulting from a second order, centered difference approximation of the spatial derivatives. Similar studies in [6], [7] and [9] are based on schemes split into hyperbolic and parabolic operators, where the eigenvalues of the constituent operators can be computed analytically.

We attempt to obtain analytical expressions of the stability limit. This is in general not possible due to the complexity of the eigenvalues and the difficulty to solve a higher degree polynomial equation for the time step. Analytical expressions of the eigenvalues are derived for the asymptotic cases low and high spatial frequencies. In the general case, the eigenvalues are computed numerically and the time step is selected with a bisection algorithm.

We compare the theoretical stability limit in 3D with the practical stability limitation in numerical simulations for a simple geometry. We find that the practical stability limit is slightly more restrictive than the one theoretically derived.

Finally, we present results from full Navier-Stokes simulations of 3D channel

flow. We illustrate a grid resolution problem and estimate the influence from artificial viscosity.

2. The equations.

The compressible Navier-Stokes equations in conservation form read

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = \frac{1}{Re} \left( \frac{\partial F_v}{\partial x} + \frac{\partial G_v}{\partial y} + \frac{\partial H_v}{\partial z} \right),$$

where $Re$ is the Reynolds number, the inviscid flux vectors are

$$Q = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E_s \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E_s + p) \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v w \\ v(E_s + p) \end{pmatrix}, \quad H = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ w(E_s + p) \end{pmatrix},$$

and the viscous flux vectors are

$$F_v = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{pmatrix}, \quad G_v = \begin{pmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \end{pmatrix}, \quad H_v = \begin{pmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \end{pmatrix}.$$

The normalisation of the equations follow [10]. Further, $\rho$ is the density of the fluid, $u, v$ and $w$ are the velocities, $E_s$ the total energy per unit volume and $p$ the pressure. The ideal gas law constitutes the relationship $E_s = p/(\gamma - 1) + (\rho/2)(u^2 + v^2 + w^2)$ where $\gamma = c_p/c_v$ is the ratio between the specific heats at constant pressure ($c_p$) and constant volume ($c_v$). The viscous stress tensor is

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \lambda(\nabla \cdot V)I + \mu(\nabla V + (\nabla V)^T) + \nabla (c^2)$$

and

$$(f_5, g_5, h_5)^T = \tau V + k/(Pr(\gamma - 1))\nabla (c^2)$$

where $V = (u,v,w)^T$, $I$ is the unit matrix, $\mu$ and $\lambda$ are the viscosity coefficients (throughout the computations we have assumed that $3\lambda + 2\mu = 0$), $k$ is the heat conductivity, $Pr$ is the Prandtl number and $c^2 = \gamma p/\rho$ is the speed of sound.