MORE NUMERICAL EVIDENCE ON
THE UNIQUENESS OF MARKOV NUMBERS

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Abstract.
A Markov triple is a solution in positive integers of the equation $x^2 + y^2 + z^2 = 3xyz$. The maximum of the triple is a Markov Number. It is conjectured (Cassels "Introduction to Diophantine Approximations") that 2 distinct Markov triples cannot share the same Markov Number. Rosen & Patterson (Math. of Comp. 25 (1971)) tested the conjecture up to $10^{10}$ by direct computation of the Markov Numbers. Modular Arithmetic was used here to carry the computation up to $10^{105}$. No duplication was found. In addition the number of Markov numbers with $N$ or less decimal digits is found to be approximately $N^2$ as in Rosen & Patterson.

1. Introduction.
Let $\Phi(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with discriminant $d(\Phi) = b^2 - 4ac > 0$. It is well-known (e.g. [1], [2]) that if $d(\Phi)$ is not a square, and if $m(\Phi)$ is the minimum of $\Phi$ for $x, y$ integers not both 0, then $\mu(\Phi) = m(\Phi)/d(\Phi)^{1/2} > 1/3$ for only a discrete sequence of forms called Markov forms. There is a one-one correspondence between those forms and the solutions in positive integers of the equation $x^2 + y^2 + z^2 = 3xyz$. Any solution $(p, a, b)$ of this equation is called a Markov triple. We will always assume that $p = \max (p, a, b)$, where $p$ is called a Markov Number. The form $\Phi$ associated with $(p, a, b)$ satisfies $\mu(\Phi) = p/(9p^2 - 4)^{1/2} > 1/3$.

The sequence of quadratic irrationals which are roots of the Markov forms play an important role in the theory of Diophantine Approximations [1].

All the Markov triples can be obtained easily in a recursive manner, starting with the triple $(1, 1, 1)$ and obtaining from each triple $(p, a, b)$ two new triples $(p_1, a, p)$ and $(p_2, p, b)$ according to the formulae:

(1) $p_1 = 3pa - b$

(2) $p_2 = 3pb - a$.

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All the Markov triples can therefore be represented as the nodes of a binary tree, where at each node the left branch is generated by formula (1) and the right one by formula (2). The first rows of the tree are shown in Figure 1 which also indicates a numbering of the rows which will be used later for reference.

$$\begin{align*}
\text{row 0} & \quad (1, 1, 1) \\
\text{row 1} & \quad (2, 1, 1) \\
\text{row 2} & \quad (5, 1, 2) \\
\text{row 3} & \quad (13, 1, 5) \quad (29, 5, 2) \\
\text{row 4} & \quad (34, 1, 13) \quad (194, 13, 5) \quad (433, 5, 29) \quad (169, 29, 2)
\end{align*}$$

Fig. 1. The Markov Tree.

It is customary to denote the Markov form corresponding to the triple \((p, a, b)\) by \(\Phi_p\). As Cassels [1] notes, “there is some ambiguity in this notation” since it is not known if two distinct Markov triples can share the same Markov Number. Cassels conjectures that such triples do not exist. Rosen and Patterson [3] computed all the Markov numbers up to \(10^{30}\) and did not uncover any duplication. Their method was a direct computation of the Markov Tree using multiple precision. The method is limited by the rapid growth of the Markov numbers requiring higher and higher precision. The purpose of this paper is to test the Cassels conjecture in a much larger range (up to \(10^{105}\)). Residue Arithmetic is used so that all computations are performed in single precision.

2. The Method.

2.1. The general idea.

The Markov tree is computed modulo some large integer \(N_0\) which assures that all numbers involved will not exceed \(N_0\). If all Markov numbers obtained are distinct mod \(N_0\) then they are distinct.

If two Markov numbers \(P_1, P_2\) are congruent mod \(N_0\), then they are computed mod a sequence of integers \(N_1, N_2, \ldots, N_k\) such that \(N_0, \ldots, N_k\) are pairwise relatively prime. If equality holds for each \(i = 0, \ldots, k\) while

$$\max(P_1, P_2) \leq \prod_{i=0}^{k} N_i,$$

we would conclude that \(P_1 = P_2\).

To implement this idea we have to overcome two difficulties: