THE CONVERGENCE OF
PRODUCT INTEGRATION RULES

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Abstract.
A recent theorem due to Nevai on the mean convergence of Lagrange interpolation is used to obtain sufficient conditions for the convergence of quadrature rules for product integration.

1. Introduction.
Two closely related methods for the approximate evaluation of integrals of the form

\[ I(f; \lambda) = \int_c^d k(\lambda; x)f(x)\,dx = \int_c^d w(x)K(\lambda; x)f(x)\,dx \]

have recently been discussed by Elliott and Paget [1] and Sloan [3]. In [1], \((c, d)\) is assumed to be a real interval; \(w\) is a non-negative weight function such that \(\int_c^d w(x)\,dx > 0\) and \(\int_c^d w(x)x^n\,dx\) exist for \(n = 0, 1, 2, \ldots\); \(K\) is a function depending on a parameter \(\lambda\) and is such that it has singularities on or close to the interval \((c, d)\), or is highly oscillatory; and \(f\) is a function which is "well-behaved" on \((c, d)\). The integral \(I(f; \lambda)\) is approximated by a quadrature sum \(Q_n(f; \lambda)\):

\[ Q_n(f; \lambda) = \sum_{j=1}^n A_{j,n}(\lambda)f(x_{j,n}) , \]

where \(x_{j,m}, j=1(1)n\) are the zeros of the polynomial \(p_m\), \(\{p_j\}\) being a sequence of polynomials orthogonal with respect to \(w\) over \((c, d)\). If \(L_n(w, f)\) denotes the Lagrange interpolation polynomial of degree \(\leq (n-1)\) which coincides with \(f\) at the points \(x_{j,m}, j=1(1)n\), then

\[ Q_n(f; \lambda) = \int_c^d w(x)K(\lambda; x)L_n(w, f; x)\,dx . \]

In the present paper we consider the convergence of \(Q_n(f; \lambda)\) to \(I(f; \lambda)\). If we set

\[ R_n(f; \lambda) = I(f; \lambda) - Q_n(f; \lambda) , \]

then, in the particular case when \((c, d)\) is finite and \(f\) is continuous on \([c, d]\) we seek conditions on \(K\) sufficient to guarantee that \(\lim_{n \to \infty} R_n(f; \lambda) = 0\). Some such
conditions have previously been given in [1]; we now obtain considerably better
ones.
Sloan [3] treats the case when \((c,d)\) is finite and, without loss of generality,
takes the interval to be \((-1,1)\). He considers the first form of \(I(f; \lambda)\) given in (1.1)
and proposes a quadrature rule of the form (1.2) where, irrespective of \(k\), the nodes
\(x_{j,n}\) are always chosen to be the zeros of the Chebyshev polynomial \(T_n\), that is \(x_{j,n} = \cos((2j-1)\pi/2n), j=1(1)n\). The convergence of such rules is discussed. In
addition, Sloan applies weighted mean convergence properties of Lagrange
interpolation to obtain convergence criteria for the rules described in [1]. In
particular he demonstrates ([3; Th. 4]) that \(\lim_{n \to \infty} R_n(f; \lambda) = 0\) for all weight
functions \(w\) and all functions \(f\) continuous on \([-1,1]\) provided that
\[
\int_{-1}^{1} w(x)K(\lambda; x)\|f(x)\|^p dx
\]
exists for some \(p > 2 - \min\{2/(2\alpha + 3), 2/(2\beta + 3)\}\).

It is the purpose of this paper to improve upon this result. In order to do so we
utilise a powerful theorem, recently published by Nevai [2], on the mean
convergence of Lagrange interpolation for a generalised Jacobi weight function.
In the following section we define such weight functions and state a particular
version of Nevai's main result which is sufficient for our purposes. The
convergence theorems then follow as a sequence of corollaries of the Nevai
theorem.

2. Sufficient Conditions for Convergence.

The Jacobi weight function \((1-x)^\alpha(1+x)^\beta, \alpha, \beta > -1,\) shall be denoted by \(w^{(\alpha, \beta)}\).

A weight function \(w\) is a generalised Jacobi weight function if, for some \(\alpha, \beta > -1,\)

(i) \(0 < A \leq w(x)/w^{(\alpha, \beta)}(x) \leq B\) for all \(x \in [-1,1]\), where \(A\) and \(B\) are constants;
(ii) \(w/w^{(\alpha, \beta)}\) is continuous on \([-1,1]\);
(iii) \(\int_0^1 \omega(\delta)\delta^{-1} d\delta\) exists, where \(\omega\) is the modulus of continuity of \(w/w^{(\alpha, \beta)}\).

Then we write \(w \approx w^{(\alpha, \beta)}\).

Nevai [2] gives the following theorem.

**Theorem.** Suppose \(w \approx w^{(\alpha, \beta)}\) and \(L_n(w, f; x)\) denotes the Lagrange interpolation
polynomial of degree \(\leq (n-1)\) which coincides with \(f\) at the zeros of \(p_n\), where
\(\int_{-1}^{1} w(x)p_j(x)p_k(x)dx = 0\) for \(j \neq k\). Then, for every function \(f\) continuous on \([-1,1],\)

\[
\lim_{n \to \infty} \int_{-1}^{1} w^{(\alpha, \beta)}(x)\|f(x) - L_n(w, f; x)\|^q dx = 0
\]