ON THE OPTIMAL APPROXIMATION OF
BOUNDED LINEAR FUNCTIONALS IN HILBERT
SPACES OF ANALYTIC FUNCTIONS

AUGUSTIN PAULIK

Abstract.
Optimal numerical approximation of bounded linear functionals by weighted
sums in Hilbert spaces of functions analytic in a circle $K_r$, in a circular annulus
$K_{r_1, r_2}$ and in an ellipse $E_r$ is investigated by Davis' method on the common
algebraic background for diagonalising the normal equation matrix. The weights
and error functional norms for optimal rules with nodes located angle-equidistant
on the concentric circle $\partial K_r$ or on the confocal ellipse $\partial E_r$ and in the interval
$[-1, 1]$ for an arbitrary bounded linear functional are given explicitly. They are
expressed in terms of a complete orthonormal system in the Hilbert space.

Key words and phrases. Optimal approximation rule, reproducing kernel function,
Gram matrix.

1. Introduction.
Let $H(B)$ denote the Hilbert space of the functions analytic in a com-
plex region $B \subset \mathbb{C}$, with the inner product $(.,.)$, the complete orthonor-
mal system $\{p_m\}_{m=0}^{\infty}$ and the reproducing kernel function
\begin{equation}
K(x, y) = \sum_{m=0}^{\infty} p_m(x) \overline{p_m(y)}, \quad x, y \in B,
\end{equation}
cf. [3, pp. 316–327]. For a bounded linear functional $Lu = (u, g)$ in $H(B)$
let us consider the approximation by the weighted sum
\begin{equation}
L_n u = \sum_{k=0}^{n-1} a_k u(x_k) = (u, \sum_{k=0}^{n-1} a_k g_{x_k}),
\end{equation}
with $n \in \mathbb{N}$, the weights $a_0, \ldots, a_{n-1} \in \mathbb{C}$ and the pairwise distinct nodes
$x_0, \ldots, x_{n-1} \in B$, where
\begin{align*}
g(x) &= \sum_{m=0}^{\infty} p_m(x) \overline{L_p m} \\
g_{x_k}(x) &= K(x, x_k), \quad k = 0, 1, \ldots, n-1
\end{align*}
are the functional representers of the functional $L$ and the point-func-
tionals $L_{x_k} u = u(x_k)$ according to the Riesz representation Theorem.

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For fixed nodes \( x_0, \ldots, x_{n-1} \) let us call the formula
\[
L_n^0 u = \sum_{k=0}^{n-1} a_k^0 u(x_k)
\]
an optimal approximation rule, if it minimizes the norm of the error functional \( E_n = L - L_n \) in the set of all formulas of the type (1.2), i.e.
\[
\|L - L_n^0\| = \|g - \sum_{k=0}^{n-1} a_k^0 g_{x_k}\| = \inf_{a_0, \ldots, a_{n-1} \in \mathbb{C}} \|g - \sum_{k=0}^{n-1} a_k g_{x_k}\|.
\]
(1.3)

It is well known that the solution of the approximation problem (1.3) exists, is unique and is given by the orthogonal projection of the functional representor \( g \in H(B) \) into the finite dimensional space \( V_n = \text{span} \{g_{x_0}, \ldots, g_{x_{n-1}}\} \subset H(B) \). This leads to the normal equations for the optimal weights \( a_0^0, \ldots, a_{n-1}^0 \):
\[
\sum_{l=0}^{n-1} a_l^0 (g_{x_l}, g_{x_k}) = (g_{x_k}, g), \quad k = 0, \ldots, n-1
\]
or briefly
(1.4)
\[
G a^0 = b
\]
where we have set
\[
a^0 = [a_0^0, \ldots, a_{n-1}^0]', \quad b = [Lg_{x_0}, \ldots, Lg_{x_{n-1}}]'
\]
and
\[
G = G(g_{x_0}, \ldots, g_{x_{n-1}}) = [(g_{x_k}, g_{x_l}); k, l = 0, \ldots, n-1]
\]
for the Gram matrix of the point functional representers. For the norm of the error functional \( E_n^0 = L - L_n^0 \) we then obtain by using the relation
\[
(g_{x_k}, g_{x_l}) = K(x_k, x_l)
\]
(1.5)
\[
\|E_n^0\|^2 = \|g\|^2 - \sum_{k=0}^{n-1} a_k^0 a_k^0 K(x_k, x_k)
\]
and hence the optimal error estimation formula:
\[
\|Lu - L_n^0 u\| \leq \|E_n^0\|\|u\|.
\]
Consequently, to compute the optimal approximation rule for a given bounded linear functional \( L \), it is sufficient to solve the system (1.4). Therefore the form of the Gram matrix \( G \) and through it the knowledge of a regular transformation of \( G \) into the diagonal matrix
\[
U_1 G U_2 = \text{diag}(d_0, \ldots, d_{n-1})
\]
are of importance, if one wants to compute the minimizing weights \( a^0 \) explicitly. These explicit formulas for \( a^0 \) may be of interest, e.g. if there is the need to choose the number \( n \) of the nodes \( x_k \) in such an order of magnitude that difficulties in the numerical inversion of the matrix \( G \)