SCIENTIFIC NOTES

AN ALGORITHM TO DECIDE
IF THE INTERSECTION OF CONVEX POLYHEDRAL
CONES HAS A NON EMPTY INTERIOR

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In an issue of this journal, Sverre Storøy [8] proposed an algorithm to find if the intersection of \( m \) convex polyhedral cones has a non empty interior or not. The problem aroused in an earlier paper [3] dealing with equilibrium prices. In the present paper we propose an alternative algorithm which turns out to be much more efficient computationally and to avoid a serious drawback present in Storøy's algorithm. We first present the general context in which the problem of the \( m \) intersecting cones arises. We then give the main steps of Storøy's algorithm. Finally we present the alternative algorithm and indicate its advantages.

The problem of computing equilibrium prices in general economic equilibrium models has altogether a long and short story. A first attempt was made by Fisher [6] more than 80 years ago. Then the computational aspects of the general equilibrium problem were left aside. Until the late sixties, research in this field concentrated on the theoretical aspects of general economic equilibrium and culminated in Debreu's classic [5] and Arrow and Hahn's textbook [1]. In the late sixties, Scarf [7] proposed a set of algorithms based on the approximation of fixed points and since then, the computation of prices has been a lively research field. Using a different approach, Boyer [2] and Boyer, Storøy and Thore [3], [4] developed an algorithm designed to compute equilibrium yields on financial markets given that each agent, here financial institution, is a portfolio optimizer.

In this model, each financial institution is assumed to maximize a linear function \( c'x \), where \( c \) is the vector of yields and \( x \) is the portfolio, subject to a set of linear constraints defining the economic environment of the institution. Each institution is characterized by its set of constraints. The algorithm then proceeds to determine endogeneously the vector \( c \).
which makes the optimal decisions by the independent institutions consistent with each other and with the financial markets as a whole. To do so, a set of linking constraints, market constraints, is added to the original sets of constraints. The first step of the algorithm is to find a set of \( n \) portfolios, one for each institution, satisfying all the constraints. The portfolio so assigned to a given institution will be optimal for that institution if the vector of yields is within a certain cone. Moreover, it will be the unique optimum if the vector of yield is interior to the cone. The algorithm therefore assigns an open cone to each institution. An equilibrium vector of yields is obtained if the intersection of those open cones has a non-empty interior. This is the interpretation of problems \( P1 \) and \( P2 \) in Storøy's paper.

It is in this context that the problem appears of finding if the intersection of \( m \) convex polyhedral cones has a non-empty interior or not. Storøy's algorithm proceeds through the polyhedral representation of the cones and a very ingenious utilization of Gordan's theorem of the alternative to define a quadratic programming problem whose solution directly reveals if the intersection has an empty interior or not. To achieve such a task, the algorithm must rule out degeneracy in the definition of the individual cones. This turns out to be a major drawback since in the original context where the algorithm was born, degeneracy is always present. The algorithm we present below does not rule out such degeneracy and uses only a linear programming framework rather than a quadratic programming one without increasing the dimensions of the problem.

Using the notation appearing in Storøy's paper, we define the \( i \)th convex polyhedral cone as \( L_i \)

\[
L_i = \{ c \in \mathbb{R}^n | c = A_i x_i, \quad \alpha_i > 0 \}
\]

where \( \alpha_i > 0 \) indicates that we are interested only in the interior of the cones in order to obtain a unique optimal decision from each agent. Our problem is to find if \( L \)

\[
L = L_1 \cap L_2 \cap \ldots \cap L_m = \bigcap_{i=1}^m L_i
\]

is empty or not and in the latter case to find a vector \( c \) in \( L \). The intersection of the \( m \) cones can be written as follows,

\[
L = \{ c \in \mathbb{R}^n | A_1 x_1 = c, \\
A_2 x_2 = c, \\
\vdots \\
A_m x_m = c, \\
\alpha_i > 0, \ i = 1, \ldots, m \}.
\]