INTERVALS OF PERIODICITY AND ABSOLUTE STABILITY OF EXPLICIT NYSTRÖM METHODS FOR $y'' = f(x, y)$

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Abstract.

We examine intervals of periodicity and absolute stability of explicit Nyström methods for $y'' = f(x, y)$ by applying these methods to the test equation $y'' = -\lambda^2 y$, $\lambda > 0$. We consider in detail general families of fourth-order explicit Nyström methods; necessary and sufficient conditions are given to characterize methods which possess non-vanishing intervals of periodicity and absolute stability. We establish closed-form expressions giving intervals of periodicity and/or absolute stability, in case these exist, for any fourth-order method. We then show that the method $M_4(1/6, 5/6)$ has the largest interval of periodicity out of all fourth-order methods; we also obtain the fourth-order method with the largest interval of absolute stability. The corresponding results for second and third-order explicit Nyström methods are also included.

1. Introduction.

For the special second-order initial-value problem:

\[ y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \]

consider an explicit Nyström method of order $s$, based on $s - 1$ function evaluations, defined by

\begin{align*}
\text{(2a)} & \quad y_{k+1} = y_k + h y'_k + h^2 \sum_{j=1}^{s-1} \alpha_j K_j + t_k(h), \\
\text{(2b)} & \quad y'_{k+1} = y'_k + h \sum_{j=1}^{s-1} b_j K_j + t'_k(h),
\end{align*}

where

\[ K_i = f\left(x_k + \alpha_i h, \ y_k + \alpha_i h y'_k + h^2 \sum_{j=1}^{i-1} \beta_{ij} K_j\right), \quad i = 1(1)s - 1, \]

and $t_k(h), t'_k(h) = O(h^{s+1})$. For the derivation of equations governing the parameters $\alpha_i, \beta_{ij}$, and $b_j$ for methods of various orders, see Hairer [1].

Stability of the classical fourth-order Nyström method was discussed by Ansorge and Törnig [2] by applying the method to the test equation:
Lambert and Watson [3] have discussed the existence of intervals of periodicity (orbital stability, in the terminology of Stiefel and Bettis [4]); Jeltsch [5] has given a characterization for linear multistep methods possessing non-vanishing intervals of periodicity by applying these to the test equation (3). It may be noted here that the classical fourth-order method of Noumerov has an interval of periodicity $(0, (6)^{\frac{1}{4}})$; the classical fourth-order Nyström method possesses no interval of periodicity, and consequently, the numerical solutions provided by it for the test equation (3) are orbitally unstable or “spiral inwards”. Houwen [6] has analysed a class of explicit Runge–Kutta–Nyström methods of orders one and two for the integration of large systems of second order differential equations arising from the semi-discretization of certain classes of hyperbolic differential equations. The methods are $m$-point ($m \geq 2$) Runge–Kutta methods, and the numerous free parameters are chosen so that the stability region along the negative axis is as large as possible. For first order methods he gives a bound for the maximum-length of the negative stability interval possible and derives some new formulas of a modified Runge–Kutta type of the same order; see also Houwen [7]. Hairer [8] has discussed unconditional stability of certain implicit Nyström methods corresponding to implicit Runge–Kutta methods based on Gaussian quadrature.

Let $Y_k = (y_k, y'_k)^T$, then at $x_k = x_0 + kh$ we may write the exact solution of (3) as

\begin{equation}
Y_{k+1} = \exp (hA) Y_k, \quad k = 0, 1, 2, \ldots, \quad A = \begin{pmatrix} 0 & 1 \\ -\lambda^2 & 0 \end{pmatrix}.
\end{equation}

Now, let $\bar{Y}_k = (\bar{y}_k, \bar{y}'_k)^T$ denote an approximation for $Y_k$ when a method defined by (2), neglecting $t_k(h)$ and $t'_k(h)$, is applied to the test equation (3). Then, we may write

\begin{equation}
\bar{Y}_{k+1} = C \bar{Y}_k, \quad k = 0, 1, 2, \ldots.
\end{equation}

For convenience, we note the following definitions adopted for methods defined by (5). In the following we set $H = \lambda h$. Also, let $r_{1,2}$ denote the eigenvalues of the matrix $C$.

**Definition 1.** An interval $(0, H_p)$ is said to be an interval of periodicity for a method (5) if, for all $H \in (0, H_p)$, $r_{1,2}$ are distinct, complex and with modulus one.

**Definition 2.** An interval $[0, H_a]$ is said to be an interval of absolute stability for a method (5) if, for all $H \in [0, H_a]$, $|r_{1,2}| \leq 1$.

While comparing stability intervals for methods of different orders $s$ ($s \geq 2$), we shall refer to “scaled” intervals of stability: $(0, H^*_p)$ for periodicity and $[0, H^*_a]$ for absolute stability, where $H^*_p = H_p/(s-1)$ and $H^*_a = H_a/(s-1)$. (See concluding remarks in Section 4.)