ON THE COLLINEATION GROUPS OF INFINITE PROJECTIVE AND AFFINE PLANES

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A partial plane is a triple $\Pi=(P,L,I)$ where $P$ is the set of points, $L$ the set of lines and $I \subseteq P \times L$ the incidence relation satisfying the axiom that

$$p_i I p_j (i,j=1,2) \implies p_1 = p_2 \text{ or } l_1 = l_2.$$ 

Using methods of E. MENDELSOHN, Z. HEDRLIN and A. PULTR we prove the following

THEOREM. Given a subgroup $G$ of the collineation group $\text{Aut } \Pi$ of some partial plane $\Pi$, there is a projective plane $\Pi'$ such that $\Pi$ is invariant under the automorphisms of $\Pi'$, $\text{Aut } \Pi' = G$, and we obtain an isomorphism of $\text{Aut } \Pi$ onto $\text{Aut } \Pi'$ by restriction. Moreover, any 3 points (lines) of $\Pi$ are collinear (concurrent) in $\Pi'$ iff they are so in $\Pi$.

Corollaries of this result improve some of E. MENDELSOHN's theorems [6,7].

1. Introduction.

In [7], E. MENDELSOHN has proved, that, given a group $G$ there exists a projective plane $\Pi$ such that for every normal subgroup $N$ of $G$ there is a line $\lambda$, such that if one takes $\lambda$ the line of infinity, the collineation group of the resulting affine plane is $N$.

One of the aims of this note is to extend this result to arbitrary subgroups $N \leq G$.

MENDELSOHN's proof uses results on automorphism groups of multicolored graphs and his construction [6] associating a projective plane with each graph such as to preserve the automorphism group.

Using a slight modification of this method and a stronger result on automorphism groups of graphs we obtain an essential improvement on Mendelsohn's results. We prove that any partial plane $\Pi$ can be imbedded in a plane $\Pi'$ such that the collineations of $\Pi'$ be extensions of the members of any specified subgroup of the collineation group of $\Pi$. (See the Theorem.) As corollaries we obtain:
COROLLARY 1. Given a group $G$ and ordinals $x_H$ for $H \leq G$ there is a projective plane $\Gamma$ and a family $\ell_{H,\alpha}$ of lines of $\Gamma$ ($H \leq G$, $\alpha < x_H$) such that the collineation group of $\Gamma$ is isomorphic to $G$, and, taking $\ell_{H,\alpha}$ for the line of infinity, the collineation group of the obtained affine plane is the subgroup corresponding to $H \leq G$. Moreover, $|\Gamma| = \max(\omega, |G|, \sum_{H \leq G} |G:H| x_H)$.

One can additionally require any of the following:

(a) all lines $\ell_{H,\alpha}$ have a point in common;
(b) no three $\ell_{H,\alpha}$ have a point in common.

Thus, in particular, the normality of the subgroups treated in [7] is superfluous.

COROLLARY 2. Given a group $G$ and a partial plane $\Pi$ there is a projective plane $\Gamma$ containing $\Pi$ such that the collineation group of $\Gamma$ is isomorphic to $G$.

This improves [6, Theorem 1] and in turn implies [6, Theorem 2]. We remark that a more essential improvement on [6, Theorem 2] will follow from [1].

2. Preliminaries.

We shall use the following definitions (cf. [6, 2]):

A partial plane $\Pi=(P,L,I)$ will be a set $P$ of points, a set $L$ of lines and an incidence relation $I \subseteq P \times L$ satisfying the single axiom

(1) $p_i \in \ell_{j,k} \ (i,j,k=1,2)$ implies $p_i = p_j$ or $\ell_i = \ell_j$.

A partial plane is finite iff $P \times L$ is finite.

A configuration is a partial plane such that either

(2) there exist 4 points no three of which are collinear, or
(3) there exist 4 lines no three of which are concurrent.

A finite configuration is said to be confined iff

(4) every point lies on at least 3 lines, and
(5) each line has at least 3 points on it.

A partial plane $\Pi=(P,L,I)$ is a partial subplane of $\Pi'=(P',L',I')$ iff $P \subseteq P'$, $L \subseteq L'$ and $I = (P \times L) \cap I'$. A subconfiguration is a partial