SOME REMARKS ON A CLASS OF RATIONAL APPROXIMATIONS TO THE COSINE

VASSILIOS A. DOUGALIS and STEVEN M. SERBIN

Abstract.

Some previously introduced methods for the numerical solution of second-order evolution equations based on a class of rational approximations to the cosine, $y$ real, with denominators $(1 + x^2 y^2)^s$ are revisited. It is shown that maximal accuracy occurs for each integer $s \geq 1$ for a suitable choice of the parameter $x$. An improved sufficient condition for unconditional stability is obtained. Conditional stability and periodicity of these methods are also studied.

1. Introduction.

The purpose of this note is to make some supplementary remarks on a class of rational approximations to $\cos y$, for real $y$, with denominator $(1 + x^2 y^2)^s$, which are especially suitable for the full discretization of linear second-order systems of ordinary differential equations, arising, for example, from semidiscretizations of second-order hyperbolic equations. The resulting two-step methods were introduced and analyzed in [2] (some of them were already considered in [11]) and were applied to hyperbolic problems in [3], [4]. These cosine methods are motivated and derived from the explicit second-order character of the equations they are intended to approximate, as opposed, for example, to methods which apply upon formulating the second-order evolution equation as a system of two first-order equations. For the latter case, Baker and Bramble [1] have introduced and analyzed single-step methods based on rational approximations to $e^y$ with denominator $(1 - x^2 z^2)^s$. While the resulting numerical schemes are different, (the single-step methods require almost double the work, as compared to the cosine methods, to achieve the same order of accuracy for a typical hyperbolic problem as was pointed out in [3]), it turns out that the approximations to the cosine may be derived also as the real part of the approximations to $e^{iy}$ of [1]. This connection enables us to use some of the elegant and powerful results of Nørsett and Wanner [9] (real pole sandwich) and Wanner, Hairer and Nørsett [12] (order stars) in order to answer some questions on the stability of the cosine methods that were left unanswered or not addressed at all in [2].

Specifically, in section 2 we recall the main points of [2] and establish the connection between the cosine methods and the single step methods of [1]. Hence,
the error of the cosine methods can be expressed in terms of the \( N \)-polynomial of [9]. In section 3, using results from [8] and [9] we basically prove that for each \( s \) there exists a value of the parameter \( x \) for which two extra orders of accuracy are obtained, thus extending the analogous result of [2] to even \( s \) as well. In section 4, using results from [9] we find a sharper lower bound for the values of \( x \) that yield unconditionally stable cosine methods. Using then a result of [5] it is seen that the methods of maximum accuracy are not unconditionally stable. We also comment on the conditional stability and the periodicity of the cosine methods.

There are still some open questions regarding the accuracy and stability of the cosine methods of which we point out one in section 5. These and the general problem of useful rational approximations to \( \cos z \) in the complex plane should probably await a systematic attack in which the program of [9], [12] can play a prominent role.

2. Preliminaries.

In [2, Prop. 3.1] the following rational approximations to \( \cos z \) are considered; for \( s \geq 1 \) integer, \( x \geq 0 \) a parameter and \( z \in C \) set

\[
(1) \quad r_s(x; z) = \left( \sum_{n=0}^{s} \varphi_n^{(s)}(x) z^{2n} \right) / \left( 1 + x^2 z^2 \right)^s.
\]

For simplicity we shall write \( r_s(x; z) = r_s(z) \), suppressing the dependence on \( x \). In (1) \( \varphi_n^{(s)}(x) \) are the polynomials, of degree \( 2 \min(n,s) \) in \( x \), given by

\[
(2) \quad \varphi_n^{(s)}(x) = \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!} \left( \begin{array}{c} s \\ n-j \end{array} \right) x^{2(n-j)} \quad \text{for integer } n \geq 0,
\]

where the usual convention \( \binom{n}{j} = 0 \) if \( n > s \) is used.

It was shown in [2, Prop. 3.1] that given \( s \) and \( x \), there exists a constant \( C = C(s, x) \) such that

\[
(3) \quad |r_s(z) - \cos z| \leq C|z|^{2s+2} \quad \text{for } |z| < x^{-1}
\]

and moreover that there exists an \( x^{(s)} > 0 \) such that for all \( x \geq x^{(s)} \)

\[
(4) \quad |r_s(y)| \leq 1 \quad \text{for all } y \geq 0.
\]

(3) implies that the resulting two-step (cosine) methods defined in [2] are, in general, accurate of order \( 2s \) for the solution of second-order problems, while (4) implies unconditional stability for the second-order problem considered in [2].

Comparing with the rational approximations to the exponential of [1] we conclude by inspection that if the polynomials \( \beta_n^{(s)}(x) \) are defined as in [1, (4.19)], then

\[
\varphi_n^{(s)}(x) = (-1)^n \beta_{2n}^{(s)}(x), \quad 0 \leq n \leq s
\]