ON THE USE OF SPLINES FOR THE NUMERICAL SOLUTION OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract.
Cubic splines on splines and quintic spline interpolations are used to approximate the derivative terms in a highly accurate scheme for the numerical solution of two-point boundary value problems. The storage requirement is essentially the same as for the usual trapezoidal rule but the local accuracy is improved from $O(h^3)$ to either $O(h^6)$ or $O(h^7)$, where $h$ is the net size. The use of splines leads to solutions that reflect the smoothness of the slopes of the differential equations.

1. Introduction.
We consider the numerical solution of the system of differential equations

$$y'(x) - f(x, y(x)) = 0,$$

with the two-point boundary conditions

$$g(y(0), y(1)) = 0,$$

where $y$, $f$ and $g$ are functions with values in $\mathbb{R}^m$, and $0 \leq x \leq 1$.

Let us subdivide the $x$ range $[0, 1]$ into $n$ equal parts, such that $h = 1/n$ and $x_i = ih$, $i = 0, 1, \ldots, n$. If we integrate the $p$th equation of the system (1.1) in the interval $[x_{i-1}, x_i]$ and denote the resulting quantity by $\varphi_{pi}(y)$ then we have

$$\varphi_{pi}(y) = y_{pi} - y_{pi-1} - \int_{x_{i-1}}^{x_i} f_p(x, y(x)) \, dx = 0,$$

where $y_{pi} = y_p(x_i)$ and $p = 1, 2, \ldots, m$; $i = 1, 2, \ldots, n$. The integral in (1.3) can be evaluated by the various numerical schemes. In this paper we will discuss one such scheme that requires the numerical derivatives of $f_p$ [1, pp. 284–285]:

$$\varphi_{pi}(y) = y_{pi} - y_{pi-1} - \frac{h}{2} (f'_{pi} + f'_{pi-1}) + \frac{h^2}{10} (f''_{pi} - f''_{pi-1})$$

$$- \frac{h^3}{120} (f'''_{pi} + f'''_{pi-1}) + O(h^7) = 0,$$

where \( f_{pi} = f_p(x_i, y(x_i)) \), \( y \) is now a vector with the components \( y_{pi} \) and the superscripts on \( f \) denote its derivatives.

From now on let us denote by \( \varphi \) and \( g \) the vectors with the components \( \varphi_{pi} \) and \( g_p \) respectively. Then equation (1.4) can be written as

\[
\varphi(y) = 0
\]

and (1.2) as

\[
g(y) = 0
\]

Clearly, \( \varphi \in \mathbb{R}^{m(n+1)} \), \( y \in \mathbb{R}^{m(n+1)} \) and \( g \in \mathbb{R}^{m} \). It is worth noting that in (1.5), \( m \) components of \( y \) can be eliminated directly by expressing them as linear functions of the rest of the components by using (1.6), provided that \( g(y) \) is a linear function of \( y \). We will assume that \( g(y) \) is linear and the necessary elimination of \( y \) components has been done so that we only have to solve (1.5).

In this paper we focus our attention on the efficient computation of two items; the required derivatives of \( f \) and the roots of the nonlinear system (1.5). To this end, we will first utilize a cubic spline on spline technique or a quintic spline to determine approximations to the first and second derivatives, and then use a nonlinear iterative technique to solve the resulting system of equations (1.5).

It is possible to compute the derivatives of \( f \) directly by differentiating the differential equation ([1], p. 246). For example

\[
f' = f_x + f_y \frac{dy}{dx} = f_x + f_y f,
\]

where the subscripts denote partial derivatives. The computation of higher derivatives of \( f \) is, in many cases, rather cumbersome. The partial derivatives can, in principle, be evaluated analytically, but more often than not, \( f \) is a complicated computer program and these computations are not possible without the expenditure of a considerable amount of work. Alternatively, it is also possible to use numerical derivatives of \( f \). This is done in [2] to solve equations of the type

\[
\varphi_{pi} = y_{pi} - y_{p,i-1} - \frac{h}{2} (f_{pi} + f_{p,i-1}) + \frac{h^3}{12} f_{p,i}^{2} + \frac{h^3}{480} f_{p,i}^{4} + O(h^7)
\]

in a deferred correction mode. The principal disadvantages of using numerical derivatives for \( f_{p,i-\frac{1}{2}} \) and \( f_{p,i+\frac{1}{2}} \) from the usual piecewise interpolating polynomials is that these polynomials not only overlap but also lack smoothness at the junctions (net points \( x_i \)). Furthermore, the deferred correction mode necessitates the repeated solution of equations of the type \( u^{1}(y^{(k)})u^{(k)} = b \) for different right hand sides \( b \). We shall have more to say about this aspect of the