Abstract.

The theory of positive real functions is used to provide bounds for the largest possible disk to be inscribed in the stability region of an explicit Runge-Kutta method. In particular, we show that the closed disk $|\zeta + r| \leq r$ can be contained in the stability region of an explicit $m$-stage Runge-Kutta method of order two if and only if $r \leq m - 1$.

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1. Introduction.

It is well known that an explicit $m$-stage Runge-Kutta method applied to the linear test equation, $y' = \lambda y$, $\lambda \in \mathbb{C}$, results in a recursion formula of the form $y_{n+1} = P(\lambda h)y_n$ where $P(\zeta)$ is a polynomial of degree $m$ and $h$ is the step size. The region of absolute stability $S_P$ for such a method is given by

$$S_P = \{\zeta \in \mathbb{C} : |P(\zeta)| \leq 1\}.$$

When designing a Runge-Kutta method one will usually have some freedom in choosing the coefficients of the method. One might wish to utilize this freedom to optimize $S_P$ in some sense. It is impossible to find an optimization criterion that would be appropriate in general, since the desired shape of the stability region will depend on the problem at hand. In this paper we consider the largest disk centered at $(-r, 0)$ with radius $r$ that can be contained in the stability region of the method. In doing this, we compromise between stretching the stability region in the real and in the imaginary direction, and in addition the problem is relatively simple to analyze.

This idea was introduced by Jeltsch and Nevanlinna [4]. They proved that the closed disk $|\zeta + r| \leq r$ can be contained in the stability region of a consistent $m$-stage explicit Runge-Kutta method if and only if $r \leq m$. This largest disk is obtained only if the stability polynomial is given by

$$P(\zeta) = (1 + \zeta/m)^m.$$
As pointed out in [5], this can be viewed as a simple consequence of Bernstein's inequality [1, p. 91] in conjunction with the following result of [5]: \( S_P \neq S_Q \) whenever \( P \neq Q \) and the two corresponding methods have the same number of stages.

The above result is clearly of great theoretical value. But it does not provide much information for the most commonly used Runge-Kutta methods, since (1) can only be the stability polynomial of an \( m \)-stage first order method. Thus it would be of interest to know the corresponding result under the additional requirement that the method be of order \( p > 1 \).

The aim of this paper is to seek this optimal disk for general \( p \). We shall see how the proof of [4] can be modified to cover the case \( p = 2 \). We also intend to illustrate to what extent the technique of [4] is applicable in the general case. This approach leads to a study of polynomials closely related to the generalized Bessel polynomials yielding upper bounds for the optimal radii. For some special cases we provide numerical values for the optimal radii and for the corresponding stability polynomial.

In order to put the problem into a precise setting, let \( \mathcal{P}_{m,p} \) be the class of polynomials

\[
P(\zeta) = \sum_{n=0}^{m} \alpha_n \frac{\zeta^n}{n!}
\]

where \( \alpha_0 = \alpha_1 = \ldots = \alpha_p = 1 \), and \( 1 \leq p < m \) (obviously, the stability polynomials of \( m \)-stage explicit Runge-Kutta methods of order \( p \) constitute a subclass of \( \mathcal{P}_{m,p} \), possibly empty due to the order barriers. Introducing the disk \( D_r = \{ \xi \in \mathbb{C} : |\xi + r| \leq r \} \), we may define

\[
\rho = \rho(m, p) = \sup_{\xi \in \mathcal{P}_{m,p}} \{ r : D_r \subset S_{\rho} \}
\]

which is our main object of interest.

We close this introduction by making the reader aware of an interesting connection between this paper and a work of Kraaijevanger [6]. One could make a comparison with the various bounds for the optimal threshold-factor \( R_{m,p} \) in [6] since some of these are quite similar to our results and since obviously \( R_{m,p} \leq \rho(m, p) \).

2. Preliminary results.

As in [4] we shall make use of the theory of positive real functions. For details on this subject in general, we recommend the survey by Dahlquist [2]. We define \( \mathbb{C}^- \) and \( \mathbb{C}^+ \) as the open left and right half-planes, respectively. Next we remind the reader that \( f(z) \) is a positive function if it is analytic in \( \mathbb{C}^+ \) and maps \( \mathbb{C}^+ \) into \( \mathbb{C}^+ \). If such a function is real-valued for real \( z \), we say that it is a positive real function. The following well-known facts [2] about positive real functions will be needed.