SOME EXPANSIONS FOR INTEGRALS WITH WEIGHT FUNCTIONS

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Abstract.

Two classes of expansions for integrals with arbitrary weight functions are derived. As one special case is obtained a generalization of Hermite's expansion. As a possible application is indicated the calculation of integrals with arbitrary weight functions.

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1. Introduction.

In a previous paper [1] we derived the expansions defined in Theorem 1.

Theorem 1. We have the following expansions (normalizing the integration range to [0, 1]).

\[\int_0^1 f(x)\,dx = \sum_{q=0}^{2n} \alpha^{(n)}_q [f^{(q)}(1) + (-1)^q f^{(q)}(0)] - \int_0^1 f^{(2n+2)}(\xi) b_{2n,2}(\xi) \,d\xi,\]

where

\[b_{2n,2}(\xi) = 2 \sum_{q=0}^{n} \frac{\alpha^{(n)}_{2q} B_{2n+2-2q}}{(2n+2-2q)!} \left[1 - \frac{B_{2n+2-2q}(\xi)}{B_{2n+2-2q}}\right],\]

and where the coefficients \(\alpha^{(n)}_q\) satisfy

\[\alpha^{(n)}_0 = \frac{1}{2}, \quad \alpha^{(n)}_{2q-1} = -2 \sum_{k=0}^{q} \frac{\alpha^{(n)}_{2k} B_{2q-2k}}{(2q-2k)!}, \quad q = 1(1)n.\]

\[\int_0^1 f(x)\,dx = \sum_{q=0}^{n} \beta^{(n)}_{2q} f^{(2q)}(\xi) \frac{1}{2} + \sum_{q=1}^{n} \beta^{(n)}_{2q-1} [f^{(2q-1)}(1) - f^{(2q-1)}(0)] + \int_0^1 f^{(2n+2)}(\xi) c_{2n,2}(\xi) \,d\xi,\]

\[1\text{ Partly performed while the author was working as corresponding fellow at CERN, Geneva, Switzerland.}

where

\[ c_{2n,2}(\xi) = -\sum_{q=0}^{n} \frac{\beta_{2q}^{(n)} B_{2n+2-2q}(\frac{1}{2})}{(2n+2-2q)!} \left[ 1 - \frac{B_{2n+2-2q}(\xi + \frac{1}{2})}{B_{2n+2-2q}(\frac{1}{2})} \right] \]

and where the coefficients \( \beta_{q}^{(n)} \) satisfy

\[ \beta_{0}^{(n)} = 1, \quad \beta_{2q-1}^{(n)} = -\sum_{k=0}^{q} \frac{\beta_{2k}^{(n)} B_{2q-2k}(\frac{1}{2})}{(2q-2k)!}, \quad q = 1(1)n. \]

For proof see [1].

In both expansions \( 2n \) unknown coefficients are related through \( n \) equations and an infinite number of expansions exists. Special cases of the expansions above are Euler–Maclaurin’s and Hermite’s ([2]) expansions, Eq. (1), and Euler’s second expansion, Eq. (4).

In the following we will generalize theorem 1 to the case of integrals having the structure

\[ I\{f\} = \int_{0}^{1} w(x)f(x)dx, \]

where \( w(x) \) is an arbitrary weight function, the only restriction being that the integral of \( w(x) \), \( x \in [0,1] \) exists.

2. Basic formulae.

Using the formula for expanding a function \( f \) in terms of Bernoulli polynomials (see Krylov [3]), we get from Eq. (7)

\[ \int_{0}^{1} w(x)f(x)dx = M_{0} f(0) + \sum_{k=1}^{q-1} \frac{M_{k}}{k!} [f^{(k-1)}(1) - f^{(k-1)}(0)] - \frac{1}{q!} \int_{0}^{1} f^{(q)}(\xi) [M_{q}(\xi) - M_{q}] d\xi, \]

where we have defined the momentum functions \( M_{k}(\xi) \) and the moments \( M_{k} \):

\[ M_{k}(\xi) = \int_{0}^{1} w(\eta) B_{k}^{*}(\eta - \xi) d\eta, \quad M_{k} = M_{k}(0). \]

Some properties of the momentum function are given below:

\[ M_{k}(\xi) = (-1)^{k} \int_{0}^{\xi} w(\eta) B_{k}^{*}(\xi - \eta) d\eta + \int_{\xi}^{1} w(\eta) B_{k}(\eta - \xi) d\eta, \]