SOME EXPANSIONS FOR INTEGRALS WITH WEIGHT FUNCTIONS

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Abstract.

Two classes of expansions for integrals with arbitrary weight functions are derived. As one special case is obtained a generalization of Hermite's expansion. As a possible application is indicated the calculation of integrals with arbitrary weight functions.

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1. Introduction.

In a previous paper [1] we derived the expansions defined in Theorem 1.

THEOREM 1. We have the following expansions (normalizing the integration range to [0,1]).

\[
\int_0^1 f(x) dx = \sum_{q=0}^{2n} \alpha_q^{(n)} \left[ f^{(q)}(1) + (-1)^q f^{(q)}(0) \right] - \int_0^1 f^{(2n+2)}(\xi) b_{2n,2}(\xi) d\xi,
\]

where

\[
b_{2n,2}(\xi) = \sum_{q=0}^{n} \frac{\alpha_{2q}^{(n)} B_{2n+2-2q}}{(2n+2-2q)!} \left[ 1 - \frac{B_{2n+2-2q}(\xi)}{B_{2n+2-2q}} \right],
\]

and where the coefficients \( \alpha_q^{(n)} \) satisfy

\[
\alpha_0^{(n)} = \frac{1}{2}, \quad \alpha_{2q-1}^{(n)} = -2 \sum_{k=0}^{q} \frac{\alpha_{2k}^{(n)} B_{2q-2k}}{(2q-2k)!}, \quad q = 1(1)n.
\]

\[
\int_0^1 f(x) dx = \sum_{q=0}^{n} \beta_{2q}^{(n)} f^{(2q)}(\frac{1}{2}) + \sum_{q=1}^{n} \beta_{2q-1}^{(n)} \left[ f^{(2q-1)}(1) - f^{(2q-1)}(0) \right] + \int_0^1 f^{(2n+2)}(\xi) c_{2n,2}(\xi) d\xi,
\]

1 Partly performed while the author was working as corresponding fellow at CERN, Geneva, Switzerland.

where

\[ c_{2n,2}(\xi) = -\sum_{q=0}^{n} \frac{\beta_{2q}^{(n)}B_{2n+2-2q}(\xi)}{(2n+2-2q)!} \left[ 1 - \frac{B_{2n+2-2q}(\xi + \frac{1}{2})}{B_{2n+2-2q}(\frac{1}{2})} \right] \]

and where the coefficients \( \beta_{q}^{(n)} \) satisfy

\[ \beta_{q}^{(n)} = 1, \beta_{2q-1}^{(n)} = -\sum_{k=0}^{q} \frac{\beta_{2k}^{(n)}B_{2q-2k}(\frac{1}{2})}{(2q-2k)!}, \quad q = 1(1)n . \]

For proof see [1].

In both expansions 2n unknown coefficients are related through n equations and an infinite number of expansions exists. Special cases of the expansions above are Euler–Maclaurin’s and Hermite’s ([2]) expansions, Eq. (1), and Euler’s second expansion, Eq. (4).

In the following we will generalize theorem 1 to the case of integrals having the structure

\[ I\{f\} = \int_{0}^{1} w(x)f(x)dx , \]

where \( w(x) \) is an arbitrary weight function, the only restriction being that the integral of \( w(x) \), \( x \in [0,1] \) exists.

## 2. Basic formulae.

Using the formula for expanding a function \( f \) in terms of Bernoulli polynomials (see Krylov [3]), we get from Eq. (7)

\[ \int_{0}^{1} w(x)f(x)dx = M_{0}^{1} f(\xi)d\xi + \sum_{k=1}^{q-1} \frac{M_{k}}{k!} \left[ f^{(k-1)}(1) - f^{(k-1)}(0) \right] - \]

\[ - \frac{1}{q!} \int_{0}^{1} f^{(q)}(\xi) [M_{q}(\xi) - M_{q}] d\xi , \]

where we have defined the momentum functions \( M_{k}(\xi) \) and the moments \( M_{k} \):

\[ M_{k}(\xi) = \int_{0}^{1} w(\eta)B_{k}^{*}(\eta - \xi)d\eta, \quad M_{k} = M_{k}(0) . \]

Some properties of the momentum function are given below:

\[ M_{k}(\xi) = (-1)^{k} \int_{0}^{\xi} w(\eta)B_{k}(\xi - \eta)d\eta + \int_{\xi}^{1} w(\eta)B_{k}(\eta - \xi)d\eta , \]