ON A CERTAIN CLASS OF LEBESGUE CONSTANTS

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Abstract.

The Lebesgue constants associated with interpolation at the extrema of the Chebyshev polynomials are investigated. It is shown that these constants are smaller than those associated with the zeros of the Chebyshev polynomials.

1. Introduction.

Given distinct points $x_0, x_1, \ldots, x_n$ and the values of some function $f$ at these points, it is well known that there exists a unique polynomial $p$ of degree at most $n$ such that $p(x_r) = f(x_r), 0 \leq r \leq n$. We may write this interpolating polynomial $p$ in the Lagrangian form

$$p(x) = \sum_{r=0}^{n} f(x_r) \xi_r(x),$$

where the polynomials $\xi_r$, the Lagrange coefficients, are given by

$$\xi_r(x) = \prod_{j=0, j \neq r}^{n} \frac{x-x_j}{x_r-x_j}.$$ 

These polynomials have the property

$$\xi_r(x_j) = \delta_r^j.$$

We will assume that $x_r \in [-1, 1], r = 0, 1, \ldots, n$, and that we wish to evaluate $p$ only at values of $x \in [-1, 1]$. This places no restrictions on the generality of what follows.

Suppose now that in (1) the values $f(x_r)$ have to be replaced by approximations $f^*(x_r)$ such that $|f(x_r) - f^*(x_r)| \leq \varepsilon, \ r = 0, 1, \ldots, n$. (This error occurs, for example, if the values $f(x_r)$ have to be approximated from $k$-figure tables, when $\varepsilon = \frac{1}{2} 10^{-k}$.) Then $p$ is replaced by the polynomial $p^*$ given by

$$p^*(x) = \sum_{r=0}^{n} f^*(x_r) \xi_r(x),$$ 

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where

\[
(4) \quad \max_{-1 \leq x \leq 1} |p(x) - p^*(x)| \leq \epsilon \max_{-1 \leq x \leq 1} \sum_{r=0}^{n} |l_r(x)|.
\]

The number

\[
(5) \quad \Lambda_n = \max_{-1 \leq x \leq 1} \sum_{r=0}^{n} |l_r(x)|,
\]

which occurs on the right side of (4), is called the Lebesgue constant associated with the set \( \{x_0, x_1, \ldots, x_n\} \). This number arises also in the following theorem which is proved in Powell [4] and Rivlin [5].

**Theorem 1.** If \( f \) is continuous on \([-1,1]\) and \( p \) is the interpolation polynomial for \( f \) based on the set \( \{x_0, x_1, \ldots, x_n\} \), then

\[
(6) \quad \max_{-1 \leq x \leq 1} |f(x) - p(x)| \leq E_n(f)(1 + \Lambda_n),
\]

where

\[
E_n(f) = \max_{-1 \leq x \leq 1} |f(x) - \bar{p}(x)|
\]

and \( \bar{p} \) denotes the unique minimax polynomial approximation to \( f \) on \([-1,1]\) of degree at most \( n \).

By virtue of (4) and (6) it is of interest to find point sets \( \{x_0, x_1, \ldots, x_n\} \) for which \( \Lambda_n \) is 'small'. We have been unable to solve the open problem of minimising (5) over all choices of \( \{x_0, x_1, \ldots, x_n\} \). Luttmann and Rivlin [2] show that if this problem has a solution, it is not unique. To discuss this further, let \( X \) denote the infinite array of nodes

\[
X: \quad x_0^{(0)} \quad x_1^{(0)} \quad x_2^{(0)} \quad \ldots
\]

\[
\quad x_0^{(1)} \quad x_1^{(1)} \quad x_2^{(1)} \quad \ldots
\]

\[
\quad x_0^{(2)} \quad x_1^{(2)} \quad x_2^{(2)} \quad \ldots
\]

\[
\quad \ldots
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\quad \ldots
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where \(-1 \leq x_0^{(n)} < x_1^{(n)} < \ldots < x_n^{(n)} \leq 1\) for each \( n = 0, 1, \ldots \). We now replace the \( x_i \) implicit in (5) by \( x_i^{(n)} \) and replace \( \Lambda_n \) by \( \Lambda_n(X) \). Erdős [1]