ON THE PRIME ZETA FUNCTION

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Abstract.

The basic properties of the prime zeta function are discussed in some detail. A certain Dirichlet series closely connected with the function is introduced and investigated. Its dependence on the structure of the natural numbers with respect to their factorization is particularly stressed.

0. Introduction.

We define the prime zeta function \( P(s) \), \( s = \sigma + i \tau \), through

\[
P(s) = \sum_{\rho} p^{-s}
\]

with the summation performed over all primes \( p \). The series converges absolutely when \( \sigma > 1 \). In the next section we shall show that \( P(s) \) can be expressed as an infinite series involving the usual Riemann zeta function. Simultaneously, this formula provides an analytic continuation to the strip \( 0 < \sigma \leq 1 \). As has been shown by Landau and Walfisz [1] the function \( P(s) \) cannot be continued beyond the line \( \sigma = 0 \). This is due to the fact that we have a clustering of singular points along the imaginary axis emanating from the non-trivial zeros of the zeta function on the critical line \( \tau = \frac{1}{2} \). On the real axis we have singular points for \( s = 1/k \) where \( k \) runs through all positive integers without a square factor (i.e. for \( s = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{10}, \frac{1}{11}, \frac{1}{13}, \ldots \)). When \( s \) is close to 1 we have (\( \varepsilon \) real, \( \varepsilon > 0 \)):

\[
P(1 + \varepsilon) = \log \frac{1}{\varepsilon} + C + O(\varepsilon)
\]

where \( C = \sum_{n=2}^{\infty} (\mu(n)/n) \log \zeta(n) \approx -0.315718452 \). Here \( \mu(n) \) is the usual Möbius function defined through

\[
\mu(n) = \begin{cases} 
1 & n=1 \\
0 & \text{if } n \text{ contains a square factor} \\
(-1)^q & \text{if } n \text{ is the product of } q \text{ different prime factors.}
\end{cases}
\]

The function \( P(s) \) has been tabulated for \( s = 2, 3, 4, \ldots \) by several authors starting with Euler; a 24 decimal table can be found in [2].
1. Basic properties of the function $P(s)$.

Starting from the well-known formula

$$
\zeta(s) = \prod_{\varphi} (1 - p^{-s})^{-1}
$$

we find:

$$
\log \zeta(s) = -\sum_{\varphi} \log (1 - p^{-s}) = \sum_{\varphi} \sum_{r=1}^{\infty} \frac{p^{-rs}}{r}
$$

$$
= \sum_{r=1}^{\infty} \frac{p^{-rs}}{r} = \sum_{r=1}^{\infty} \frac{P(rs)}{r}
$$

Using Möbius' inversion formula we obtain the important relation:

$$
P(s) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \zeta(ks). \tag{1.2}
$$

This formula seems first to have been given by Glaisher [3]. It can be used for numerical computation for all except the singular points in the right half plane. The computational difficulties increase tremendously as we approach the imaginary axis, especially for large values of $\tau$. In the numerical computation the straight-forward technique described by Haselgrove–Miller [4] has been used. Essentially, we apply the following semi-convergent series, suitably truncated:

$$
\zeta(s) = \sum_{n=1}^{N} n^{-s} + N^{-s} \left\{ \frac{N}{s-1} - \frac{1}{2!} \frac{B_1}{N} - \frac{B_2}{4!} \frac{(s+1)(s+2)}{N^3} \right. \\
+ \frac{B_3}{6!} \frac{(s+1)(s+2)(s+3)(s+4)}{N^5} - \left. \ldots \right\} \tag{1.3}
$$

The logarithm is defined so that $\log \zeta(s)$ is real when $s$ is real and $> 1$; further we claim that $P(s)$ be continuous everywhere except in the singular points.

Very little is known with regard to the roots of the equation $P(s) = 0$. The following four roots, together with their complex conjugates, which are fairly far away from the imaginary axis, have been found together with one root of $P(s) = -1$ and one (real) root of $P(s) = 1$ (table I below).