APPROXIMATION BY NON-NEGATIVE
ALGEBRAIC POLYNOMIALS

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Abstract.

Theorem 1 gives an estimate for the approximation of a continuous function $f$ by polynomials resulting from the convolution of $f$ with non-negative algebraic polynomials $p_n$. Jackson's theorem can be deduced from it by choosing a particular $p_n$ whose second Chebyshev–Fourier coefficient is sufficiently close to $-1$.

The classical theorem of Jackson [1, p. 15] states that if $f \in C[a,b]$ and $u_f(\delta)$ is its modulus of continuity, then for each positive integer $n$ there is an algebraic polynomial $p_n \in \mathcal{P}_n$ (the class of all polynomials of degree $\leq n$) such that, in the uniform norm,

$$\|f - p_n\| \leq C u_f(1/n)$$

where $C$ is a constant independent of $f$ and $n$. DeVore [2] has given a direct proof of (1), by considering convolution of $f$ with non-negative algebraic polynomials. Though the approximating polynomial is of degree $4n-4$, his value of $C$ is much larger than that indicated by the usual trigonometric polynomial approach, viz., $C = 1 + \pi^2/2 < 6$; see, for example, Rivlin [3], pp. 21–22. Moreover, his proof makes use of the order of approximation obtained in the approximation of functions of class Lip 1 ([2], Proposition 1). More recently, Bojanic and DeVore [4] have given a similar proof which is essentially based on the result of Shisha and Mond [5] on the order of convergence of linear positive operators.

The purpose of the present paper is to give a simple proof based on more elementary arguments.

Let $p_n(x) \in \mathcal{P}_n$ be such that $p_n(x) = p_n(-x)$ and $p_n(x) \geq 0$ on $[-1,1]$. If

$$\int_{-1}^{1} p_n(x) dx = c_n, \quad c_n > 0$$

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set \( P_n(x) = (1/c_n)p_n(x) \) and define \( L_n(f; p_n; x) \in P_n \) by

\[
L_n(f; p_n; x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)P_n(t-x)\,dt
\]

(it will be assumed throughout in the following that \( n \) is a positive even integer). If \( T_n(x) = \cos(n \arccos x) \) denotes the Chebyshev polynomial of degree \( n \) on \([-1, 1]\) then we may write

\[
p_n(x) = \sum_{k=0}^{n/2} c_{n,k}T_k(x).
\]

Our main result is

**Theorem 1.** If \( f \in C[-\frac{1}{2}, \frac{1}{2}] \) and \( f(-\frac{1}{2}) = f(\frac{1}{2}) = 0 \), then for \( L_n(f; p_n; x) \in P_n \) defined by (2) and for every positive integer \( m \),

\[
||f(x) - L_n(f; p_n; x)|| \leq w_f(1/m)(\frac{3}{2} + m \cdot \beta_n)
\]

where

\[
\beta_n^2 = (1 + c_{n,1})(1 - c_{n,1})
\]

The proof is based on the following

**Lemma 1.** Let \( f \) satisfy the conditions of Theorem 1. Let \( V(x) \) denote the total variation of \( \alpha(x) \), and let \( |\alpha(x)| \) be monotonically increasing on \([-\frac{1}{2}, \frac{1}{2}]\). Then for every positive integer \( m \),

\[
\left| \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,d\alpha(x) \right| \leq w_f(1/m) \left( V(\alpha) + m \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} |\alpha(x)|\,dx \right).
\]

**Proof.** For a positive integer \( m \), consider the partition of \([-\frac{1}{2}, \frac{1}{2}]\) by the points \( x_k = -\frac{1}{2} + k/m, \ k = 0(1)m. \) Define

\[
R_m^+(f; \alpha) = \sum_{k=1}^{m} f(x_k)\Delta \alpha_k, \quad \Delta \alpha_k = \alpha(x_k) - \alpha(x_{k-1}).
\]

Since \( \alpha(x) \in BV[-\frac{1}{2}, \frac{1}{2}] \), \( \int_{x_{k-1}}^{x_k} d\alpha(x) = \Delta \alpha_k, \ k = 1(1)m \), therefore

\[
\left| \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,d\alpha(x) - R_m^+(f; \alpha) \right| = \left| \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} (f(x) - f(x_k))\,d\alpha(x) \right| \leq w_f(1/m)V(\alpha)
\]

since \( \int_{-\frac{1}{2}}^{\frac{1}{2}} d\alpha(x) = V(\alpha) \). Thus