APPORXIMATION OF CONVEX DATA

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Abstract.
A method is described of obtaining convex polynomial approximations to discrete sets of convex data. The approximations are best convex combinations of certain component convex functions. A Weierstrass-type theorem is proved, to justify the choice of component functions, and numerical illustrations are given.

1. Introduction.
One is sometimes presented with a discrete set of data which is convex, or where physical reasons suggest that the data would be convex but for experimental error. In these circumstances, it seems unsatisfactory to use the standard least squares method, which may produce approximations with undesired inflexions. It would appear preferable to make use of our knowledge of the convexity of the data, in order to produce better approximations to the underlying convex function, say $g(x)$, and its first few derivatives, assuming that these exist and are required.

Let us suppose that $g(x)$ is convex on a finite interval $[a, b]$. If we choose some sequence of functions $\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots$, which are convex on $[a, b]$, we may consider using functions of the form

$$\Phi_n(x) = \sum_{j=0}^{n} c_j \varphi_j(x), \quad c_j \geq 0,$$

for approximating to $g(x)$. In (1), since we are using a sum of non-negative multiples of the component functions $\varphi_j(x)$, we have that $\Phi_n(x)$ also is convex on $[a, b]$. It still remains to determine two things: what Rice [4] calls the "norm and form" of the approximation. The first of these, the choice of norm, will decide on the "best" values for the coefficients $c_j$ in (1). The second task is to make a suitable choice of the component functions $\varphi_j(x)$, which is more difficult.

At this stage, we may point out one deficiency of this approach to the problem. Our method will not always find the best (in any sense)
convex polynomial approximation of given degree to an arbitrary set of data. For to guarantee this, we would require every convex polynomial to be expressible as a convex combination of the $v_j(x)$, which, in the following section, we choose to be certain convex polynomials. That is, considering second derivatives, we would require every polynomial which is non-negative, say on $[0,1]$, to be expressible as a sum of non-negative multiples of the polynomials $v_j''(x)$. It is impossible to satisfy this requirement for any choice of non-negative polynomial component functions. For suppose that $v_j(x)$, $j=0,1,\ldots,n$, is any set of polynomial component functions with degree not exceeding $n$ and that each $v_j(x) \geq 0$ for $x \in [0,1]$. Let us choose a point $\alpha$, $0 < \alpha < 1$, such that $v_j(\alpha) > 0$ for $j=0,1,\ldots,n$. Such an $\alpha$ exists, because the number of zeros of the polynomials $v_j(x)$ is finite. It follows that we cannot express the polynomial $(x-\alpha)^2$, for example, as a sum of non-negative multiples of the $v_j(x)$.

Despite this drawback, we will see that it is possible to make a choice of convex polynomials $v_j(x)$ which have useful convergence properties. Let us recall why polynomials have been so extensively and successfully used, particularly for approximating to discrete data. This is partly because polynomials are easily evaluated. However, the main reason, as proved by Weierstrass' theorem, is that linear combinations of the monomials $x^j$ are good enough for approximating arbitrarily closely to any continuous function on a finite interval. What we require here is a sequence of convex functions $v_j(x)$ for which we can state a Weierstrass type of theorem for convex functions. We would wish any convex function $g(x)$ to be approximable with arbitrary accuracy, on a finite interval, by a sum of non-negative multiples of the convex functions $v_j(x)$.

Later in this paper, we quote some numerical examples, which are derived in an objective way. The results seem to support the view that the method to be described here for approximating to convex data tends to produce only slightly better approximations to the function, but very much better approximations to the derivatives, than the conventional least squares method.

2. Choice of the component convex functions.

If $g(x)$ is convex on $[0,1]$, so also is its Bernstein polynomial,

$$B_n(g; x) = \sum_{j=0}^{n} \binom{n}{j} x^j (1-x)^{n-j} g(j/n),$$

for any $n=0,1,2,\ldots$. This would encourage us to use Bernstein polynomials for approximating to convex functions, were it not for a knowl-