PRODUCT TYPE QUADRATURE FORMULAS

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Abstract.

This paper is concerned with the numerical approximation of integrals of the form \( \int_a^b f(x)g(x) \, dx \) by means of a product type quadrature formula. In such a formula the function \( f(x) \) is sampled at a set of \( n + 1 \) distinct points and the function \( g(x) \) at a (possibly different) set of \( m + 1 \) distinct points. These formulas are a generalization of the classical (regular) numerical integration rules. A number of basic results for such formulas are stated and proved. The concept of a symmetric quadrature formula is defined and the connection between such rules and regular quadrature formulas is discussed. Expressions for the error term are developed. These are applied to a specific example.

1. Introduction.

Most numerical integration formulas have the form

\[
\int_a^b f(x) \, dx \approx \sum_{i=0}^n c_i f(x_i).
\]

The error in (1.1) is

\[
E(f) = \int_a^b f(x) \, dx - \sum_{i=0}^n c_i f(x_i).
\]

We will call formulas of type (1.1) regular quadrature formulas.

In this paper we place our attention on integrals of the following type

\[
\int_a^b f(x)g(x) \, dx.
\]

The form of the numerical integration formulas considered is

\[
\int_a^b f(x)g(x) \, dx \approx (f(x_0), \ldots, f(x_n))\begin{pmatrix}
  a_{00} & a_{01} & \cdots & a_{0m} \\
  a_{10} & a_{11} & \cdots & a_{1m} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n0} & a_{n1} & \cdots & a_{nm}
\end{pmatrix}\begin{pmatrix}
  g(y_0) \\
  g(y_1) \\
  \vdots \\
  g(y_m)
\end{pmatrix}.
\]

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For notational convenience we define $f(x_0, \ldots, x_n) = (f(x_0), \ldots, f(x_n))^T$, $g(y_0, \ldots, y_m) = (g(y_0), \ldots, g(y_m))^T$, and $A = (a_{ij})$ the $(n+1) \times (m+1)$ matrix in (1.3). Whenever no confusion will result we will use $f$ and $g$ instead of $f(x_0, \ldots, x_n)$ and $g(y_0, \ldots, y_m)$.

The error for (1.3) is

$$E(f, g) = \int_a^b f(x)g(x) \, dx - f^T A g.$$  

The expression $E(f, g)$ is easily shown to be a bilinear functional. A quadrature formula of type (1.3) will be called a product quadrature formula.

Note that if in (1.3) $m = n$, $x_i = y_i$, and $a_{ij} = 0$ for $i \neq j$ the product quadrature formula (1.3) reduces to the regular formula of (1.1). Hence the class of product formulas includes the class of regular formulas as a subset. Throughout this paper we assume the interval $[a, b]$ is finite and the numbers $(x_0, \ldots, x_n)$ to be real and distinct, with the same being true for $(y_0, \ldots, y_m)$.

The work is organized as follows: In Section 2 we introduce the idea of a product interpolatory quadrature formula. Several basic results for such formulas are stated and proved. Four examples are exhibited in Section 3. Symmetric quadrature formulas are defined in the fourth section and the connection between such rules and regular quadrature formulas is discussed. Section 5 is devoted to the development of expressions for the error term $E(f, g)$ for three different continuity assumptions on $f$ and $g$. Section 6 gives specific error analysis for the product trapezoidal rule. The last section consists of a discussion of possible applications.

2. Basic Definitions and Elementary Lemmas.

A regular quadrature formula is said to be an interpolatory type formula whenever it is derived by integrating the polynomial $p_n(x)$ of degree less than or equal to $n$ which interpolates $f(x)$ at the $n + 1$ distinct points $x_0, x_1, \ldots, x_n$. It is well known (see [1] or [2]) that regular interpolatory type quadrature formulas are uniquely given in terms of the interval of integration and the interpolating points.

Let $\mathcal{P}_k$ be the set of all polynomials of degree $k$ or less.

**Definition 2.1.** A product quadrature formula is said to be a product interpolatory formula whenever it is derived by integrating $p_n(x)q_m(x)$, where $(p_n, q_m) \in \mathcal{P}_n \times \mathcal{P}_m$, $p_n(x)$ interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$, and $q_m(x)$ interpolates $g(x)$ at $y_0, y_1, \ldots, y_m$. 