STRICT ESTIMATION OF
THE MAXIMUM OF A FUNCTION OF ONE VARIABLE

TORSTEN STRÖM

Abstract.

We describe an algorithm for the strict estimation of \( \max_{x \in I} f(x) \) when \( f \) is analytic on \( I \). If roundoff is neglected the error bound can be made arbitrarily small by an adaptive implementation. A majorant concept is utilized and an Algol program is given. Finally we outline the special application to the strict estimation of the range of a polynomial without any determination of the zeros of the derivative.

Keywords: Optimization, Maximization, Error Bound, Range Determination.

Introduction.

In some situations a demand for strictness in an estimate may dominate over the accuracy requirements, at least in the sense that one is willing to accept a greater amount of work for a given strict bound on the error. A non-trivial problem of this type appears when we are given an approximate solution of a system of ordinary differential equations and wish to estimate the local error strictly. Given the norm in which we compute the error this becomes a problem of estimating the maximum of the absolute value of a function in one variable over an interval (Ström [6]). We believe that the desire for strictness is recognized in several situations. Of course strictness should be obtained in a global sense, i.e. we must not get trapped in a local maximum. Local properties, however, may be used as guidelines and one possible approach is to utilize the positions of zeros of \( f' \). We will take a different attitude and use a majorant concept which in a general case is not trivial but in many cases simply mechanizable.

There exist a number of methods for the efficient determination of \( \max_{x \in I} f(x) \). In fact this is often regarded as a fairly trivial special case of optimization of functions of several variables. To the best of our knowledge, however, few methods claim to work with guaranteed error bounds. Also the local-global dilemma is generally not eliminated. A notable exception is the scheme originally proposed by Moore [4] utilizing the technique of interval arithmetic. Recent work (Dussell [3])

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shows how these fundamental ideas can be implemented to provide completely rigorous bounds if interval arithmetic is applied also on the roundoff level. In fact a programming language containing interval numbers such as Triplex-Algol 60 (Apostolatos [1]), would cope perfectly with the problems of roundoff also in our scheme. We will essentially neglect roundoff in the following. We also assume that $f$ is majorizable in our sense (Ström [5]), i.e. that an absolutely (or completely) monotonic function $F$ exists such that $F \pm f$ are again absolutely (completely) monotonic. (A function $F$ is absolutely (completely) monotonic on $I$ if $F^{(v)}(x) \geq 0$ $((-1)^v F^{(v)}(x) \geq 0)$ for $x \in I$ and $v = 0, 1, \ldots$). Ström [5] discusses how such an $F$ may be automatically and efficiently constructed by a computer when $f$ is composed of elementary functions. Dahlquist [2] outlined the general idea of our scheme in a particular application. This is a modified and more complete treatment.

**Basic Algorithm.**

The fundamental observation is almost trivial and we quote it as a theorem from Dahlquist [2] without proof.

**Theorem 1.** Let $x_0 < x_1 < \ldots < x_n$ be $m+1$ distinct points in $I = [x_0, x_n]$. Assume that $G^{(m)}(x)$ is of constant sign in $I$. Let $L_m(G,x)$ be the value in $x$ of the unique polynomial of degree $\leq p - m$ interpolating to $G$ in the points $x_0, x_1, \ldots, x_n$. Then, for $x \in I$,

\[
L_{m-1}(G,x) < G(x) < L_m(G,x)
\]

(or reversed inequalities).

We will subsequently use the notations

\[
A^m_0(G,x) = \frac{1}{2}(L_{m-1}(G,x) + L_m(G,x))
\]

and

\[
E^m_0(G,x) = \frac{1}{2}(L_m(G,x) - L_{m-1}(G,x)).
\]

Now let $f$ be a given function majorized by $F$ on $I$. Then (Dahlquist [2] or Ström [5])

\[
|f(x) - A^m_0(f,x)| \leq E^m_0(F,x).
\]

Let $x_i, i = 0(1)m$, be equidistant. It is a matter of simple algebra to show that for any function $g$ with $g_i = g(x_i)$

\[
|E^m_0(g,x)| \leq \frac{1}{2} |\Delta^m g_0|
\]