SCIENTIFIC NOTES

BIASED RATIONAL CHEBYSHEV APPROXIMATION

CHARLES B. DUNHAM

Computer Science Department, University of Western Ontario, London, Ontario N6A 5B9, Canada

Abstract.

Chebyshev approximation on an interval [α, β] by ordinary rational functions when positive deviations (errors) are magnified by a bias factor is considered. This problem is related to one-sided Chebyshev approximation for large bias factors. Best approximations are characterized by alternation. Non-degenerate best approximations can be determined by the Remez algorithm. A variant of the Fraser-Hart-Remez algorithm is implemented.

Let [α, β] be a closed finite interval and \( C[α, β] \) be the space of continuous functions on [α, β] with norm

\[
\|g\| = \max \{|g(x)|: α ≤ x ≤ β\}.
\]

Let \( H_n \) be the set of polynomials of degree ≤ \( n \). Let \( R^m_\alpha[x, β] \) be the family of ratios \( p/q, p ∈ H_n, q ∈ H_m, q(x) > 0 \) for \( α ≤ x ≤ β \). Let \( s \) be a positive element of \( C[α, β] \), an ordinary multiplicative weight function. Let the bias factor \( r \) be a real number in \( (0, ∞) \) and define

\[
d_r(y) = \begin{cases} y, & y ≤ 0 \\ ry, & y > 0. \end{cases}
\]

The bias factor \( r \) is used to magnify positive deviations (errors) by a factor \( r \). The problem of \( r \)-biased \( s \)-weighted Chebyshev approximation is, given an element \( f \) of \( C[α, β] \), to find an element \( p^*/q^* \) of \( R^m_\alpha[x, β] \) to minimize

\[
e_r(p/q) = \|d_r(s(f-p/q))\| = \|sd_r(f-p/q)\|
\]

over \( p/q \in R^m_\alpha[x, β] \). Such an element \( p^*/q^* \) is called best to \( f \).

Biased linear \( L_1 \) approximation is studied by Young and Kiountouzis [8].

By taking \( r \) large, the problem becomes close to that of one-sided approximation and it has been shown by the author [5] that the uniform limit of \( r \)-biased approximations is the best one-sided approximation if the best one-sided approximation is of maximum degree.

Received February 2, 1981.
The family $R_n^l[x, \beta]$ is known to be varisolvent. If $p/q = 0$, $p/q$ has degree $l + 1$ with defect (degeneracy) $m$. If $p/q \neq 0$, we can assume $p$ and $q$ are relatively prime. Then $p/q$ has degree $l + m + 1 - d$, where the defect (degeneracy) $d = \min \{l - \hat{\epsilon}p, m - \hat{\epsilon}q\}$, where $\hat{\epsilon}$ denotes exact degree.

The problem of this paper is that of ordinary rational approximation with respect to the generalized weight function

$$w(x, a, b) = \begin{cases} rs(x)(b - a) & b - a < 0 \\ s(x)(b - a) & b - a \geq 0. \end{cases}$$

Approximation with respect to generalized weight functions is covered by the author in [3]. We obtain an analogue of the lemma of de la Vallée-Poussin [3, Lemma 2] and an alternation theorem.

**Theorem:** Let $p/q$ have degree $n$. Then $p/q$ is best to $f$ if and only if $\mu (f - p/q) = d_r (f - p/q)$ alternates $n$ times on $[x, \beta]$. A best approximation is unique.


The analogue of the Remez algorithm for our problem is

(i) Choose a trial alternant $X^0 = \{x_0^0, \ldots, x_{l+m+1}^0\}$ and set $k = 0$.

(ii) Solve the system of levelling equations

$$s(x_i) \mu (f(x_i) - p^k(x_i)/q^k(x_i)) = (-1)^k \lambda_k, \quad i = 0, \ldots, l + m + 1$$

for $p^k$, $q^k$, $\lambda_k$.

(iii) Find new alternating extrema $X^{k+1}$ of $\mu (f - p^k/q^k)$ on $[x, \beta]$ by the Remez $l + m + 2$ point simultaneous exchange, described by Meinardus [7, 107–108].

(iv) Add 1 to $k$ and go to (ii).

This algorithm is infinite. We can stop when the error curve is sufficiently levelled by the de la Vallée-Poussin type results cited earlier.

Now let us consider implementation of the algorithm. As in the case of ordinary Chebyshev approximation, solution of the levelling equations can be difficult. An additional difficulty in biased approximation is that we need to know signs of the error to solve (1) and the signs are not known a priori.

Let us drop the $k$'s in (1) for simplicity. First suppose $f - p/q$ is $< 0$ on $x_0$. Then we have

$$s(x_i) [f(x_i) - p(x_i)/q(x_i)] = -\lambda, \quad i \text{ even}$$

$$s(x_i) [f(x_i) - p(x_i)/q(x_i)] = +\lambda, \quad i \text{ odd}$$