MODIFIED CLENSHAW-CURTIS METHOD FOR THE COMPUTATION OF BESSEL FUNCTION INTEGRALS

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Abstract.

The numerical evaluation of Bessel function integrals may be difficult when the Bessel function is rapidly oscillating in the interval of integration. In the method presented here, the smooth factor of the integrand is replaced by a truncated Chebyshev series approximation and the resulting integral is computed exactly. The numerical aspects of this exact integration are discussed.

1. Introduction.

We consider the numerical computation of

\[ I = \int_0^1 f(x) J_v(ax) \, dx \]

where \( J_v(x) \) is the Bessel function of the first kind and of order \( v \), and where \( a \) is an arbitrary positive real number. An important example of this type of integral is the Fourier-Bessel transform of a function \( g(x) \):

\[ \int_0^1 g(x) x J_v(\alpha_n x) \, dx \]

where \( 0 < \alpha_1 < \alpha_2 < \ldots \) are the zeros of \( J_v(x) \).

If \( a \) is large, the integrand of (1) is rapidly oscillatory. Numerical integration is difficult and special methods should be used, for example integration between the zeros, possibly in combination with a convergence accelerating method (see Davis and Rabinowitz [9], Longman [19], Piessens [22]).

In this paper we present another method, which is especially useful if \( f(x) \) is a smooth function. This method is based on the approximation of \( f(x) \) by a truncated series of shifted Chebyshev polynomials.

Received October 27, 1982. Revised February 11, 1983.
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\begin{equation}
(3) \quad f(x) \simeq \sum_{k=0}^{N} c_k T_k^*(x), \quad 0 \leq x \leq 1.
\end{equation}

Here, the symbol \( \sum' \) indicates that the first term in the sum must be halved. The idea of applying Chebyshev polynomials for the integration is due to Clenshaw and Curtis [6].

For the computation of the coefficients \( c_k \) in (3) several good algorithms are available (see Gentleman [16], Branders and Piessens [4]).

The integral in (1) can now be approximated by

\begin{equation}
(4) \quad I \simeq \sum_{k=0}^{N} c_k M_k(a, v)
\end{equation}

where

\begin{equation}
(5) \quad M_k(a, v) = \int_{0}^{1} J_v(a x) T_k^*(x) dx, \quad k = 0, 1, \ldots
\end{equation}

are the so-called modified moments (Gautschi [15]). The computation of \( M_k(a, v) \) is the main subject of this paper.

2. A recurrence relation for \( M_k(a, v) \).

In this section we prove that the modified moments satisfy the following recurrence relation for \( k = 4, 5, 6, \ldots \)

\begin{equation}
(6) \quad \frac{a^2}{16} M_{k+4}(a, v) + \left[ (k+3)^2 - v^2 - \frac{a^2}{4} \right] M_{k+2}(a, v) + (4v^2 + 2k + 4)M_{k+1}(a, v)
\end{equation}

\begin{equation}
- \left[ 2k^2 - 6 + 6v^2 - \frac{3a^2}{8} \right] M_k(a, v) + [4v^2 - 2k + 4] M_{k-1}(a, v)
\end{equation}

\begin{equation}
+ \left[ (k-3)^2 - v^2 - \frac{a^2}{4} \right] M_{k-2}(a, v) + \frac{a^2}{16} M_{k-4}(a, v) = 0.
\end{equation}

Indeed, (5) can be written as

\begin{equation}
(7) \quad M_k(a, v) = \frac{1}{2} \int_{-1}^{1} J_v((x + 1)a/2) T_k(x) dx
\end{equation}

where \( T_k(x) \) is the Chebyshev polynomial of degree \( k \).

Let

\begin{equation}
(8) \quad K_1 = 4 \int_{-1}^{1} (1 + x)^2 (1 - x)^2 J'_v((x + 1)a/2) T_k(x) dx
\end{equation}