ON THE RESOLVENT CONDITION IN THE KREISS MATRIX THEOREM*

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Abstract.

The Kreiss Matrix Theorem asserts the uniform equivalence over all \( N \times N \) matrices of power boundedness and a certain resolvent estimate. We show that the ratio of the constants in these two conditions grows linearly with \( N \), and we obtain the optimal proportionality factor up to a factor of 2. Analogous results are also given for the related problem involving matrix exponentials \( e^{At} \). The proofs make use of a lemma that may be of independent interest, which bounds the arc length of the image of a circle in the complex plane under a rational function.

AMS Subject Classification: primary 39A11; secondary 15A45, 30A10.

1. Introduction.

Let \( A \) be an \( N \times N \) matrix that satisfies the power boundedness condition

\[
p(A) = \sup_{n \geq 0} \| A^n \| < \infty,
\]

where \( \| \cdot \| = \| \cdot \|_2 \). By a power series expansion it is readily verified that \( A \) then also satisfies the resolvent condition

\[
r(A) = \sup_{|z| > 1} (|z| - 1) \| (zI - A)^{-1} \| < \infty,
\]

and moreover \( r(A) \leq p(A) \). One of the assertions of the Kreiss Matrix Theorem [3, 4, 7] is that the converse is also valid: if \( r(A) < \infty \), then \( p(A) < \infty \) also, and \( p(A) \) can be bounded in terms of \( N \) and \( r(A) \) but otherwise independently of \( A \).

* Research supported by NSF Mathematical Sciences Postdoctoral Fellowships, by the Courant Institute of Mathematical Sciences, and by the National Aeronautics and Space Administration under Contract No. NAS1-17070 while the authors were in residence at the Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23665.

Received August 1983.
This result is useful in proofs of stability theorems for finite difference approximations to partial differential equations.

In this note we resolve an old question contributed to most recently by Tadmor [8]: given $N$ and $r(A)$, how large can $p(A)$ be? According to Tadmor, Kreiss's original proof in [4] unwinds to give a far from sharp bound

$$p(A) \lesssim [r(A)]^{N^2}, \quad (\forall A)$$

which subsequent improvements by Morton, Strang, and Miller lowered to

$$p(A) \lesssim 6^N(N + 4)^5 N r(A), \quad N^5 r(A), \quad e^{9N^2} r(A) \quad (\forall A).$$

A few years ago Strang (private communication) observed that a paper of Laptev [5] implicitly derives a much more reasonable estimate [3]

$$p(A) \lesssim (32e/\pi) N^2 r(A) \quad (\forall A).$$

Finally Tadmor's proof, which makes use of an elegant Cauchy integral argument adapted from Laptev, yields a bound that is linear in $N$,

$$p(A) \lesssim (32e/\pi) N r(A) \quad (\forall A). \quad (3)$$

Tadmor conjectures that a linear dependence as in (3) is the best possible. However, up to now the strongest growth of $p(A)$ with $r(A)$ attained by an example has been logarithmic, i.e., $p(A) \approx r(A) \log N$ [6].

First we will show that Tadmor's conjecture is correct, by exhibiting a family of matrices $\{A_N\}$ for which $p(A_N) \sim e N r(A_N)$ as $N \to \infty$. By refining the Cauchy integral argument, we will then show that for arbitrary matrices (3) can be sharpened to $p(A) \leq 2e N r(A)$. (Our proof is essentially Tadmor's, but gains the factor $16/\pi$ over his by dealing with complex functions directly rather than taking real and imaginary parts.) Together these results establish that $eN$ is the optimal constant of proportionality relating $p(A)$ to $r(A)$ except for a possible factor of 2. The final section will prove analogous results for the continuous problem involving matrix exponentials $e^{At}$.

2. Example with $p(A_N) \sim e N r(A_N)$.

Consider the $N \times N$ Jordan matrix

$$A = A_N = \begin{bmatrix}
0 & \gamma & & \\
0 & 0 & \ddots & \\
& & \ddots & \gamma \\
& & & 0
\end{bmatrix}$$