ON COMPUTING INV BLOCK PRECONDITIONINGS FOR
THE CONJUGATE GRADIENT METHOD *

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Abstract.

The INV(k) and MINV(k) block preconditionings for the conjugate gradient method require
generation of selected elements of the inverses of symmetric matrices of bandwidth 2k + 1.
Generalizing the previously described k = 1 (tridiagonal) case to k = 2, explicit expressions for the
inverse elements of a symmetric pentadiagonal matrix in terms of Green's matrix of rank two are
given. These expressions are found to be seriously ill-conditioned; hence alternative computational
algorithms for the inverse elements must be used. Behavior of the k = 1 and k = 2 preconditionings
are compared for some discretized elliptic partial differential equation test problems in two
dimensions.

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ization, iterative methods, preconditioning, sparse matrices.

1. Introduction.

In a recent paper, block incomplete Cholesky factorization was investigated
as a preconditioning for the conjugate gradient method [5]. Several variants
were introduced, of which the preconditioning INV(k) (and its modified form
MINV(k)) gave particularly encouraging results on discretized boundary-value
test problems for some self-adjoint, two-dimensional, linear elliptic partial
differential equations. If the approximating linear system has coefficients that
form a symmetric, positive-definite matrix that is also a diagonally-dominant
M-matrix, which is the generic model case, then carrying out of the incomplete
factorization can be guaranteed as well as the subsequent convergence of the
conjugate gradient iteration.

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In [5] detailed investigation of the preconditionings \( \text{INV}(k) \) and \( \text{MINV}(k) \) was limited to the case \( k = 1 \), corresponding to keeping only the principal tridiagonal portion of inverses of tridiagonal matrices arising in the course of the incomplete decomposition. This case possesses a special attraction because the inverse elements of the tridiagonal matrices can be expressed explicitly in a simple manner as the outer product of two vectors.

Here we consider extension to the case \( k \) greater than one and corresponding matrices of larger bandwidth, focusing attention on the case \( k = 2 \). The expression for the inverse elements of pentadiagonal matrices arising for \( k = 2 \) analogous to the explicit outer-product expression for \( k = 1 \) is given. However, this expression is shown to have serious shortcomings for numerical computation. An alternative computational procedure for building up the principal diagonals of the inverse is therefore required. Results of numerical experiments are given to illustrate the relative efficiency of the \( k = 1 \) and \( k = 2 \) preconditionings.

2. Incomplete block Cholesky factorization.

Incomplete block Cholesky preconditioning for the conjugate gradient method was introduced by R. R. Underwood [6] and developed and investigated in [5] and elsewhere [3]. We consider solving iteratively the symmetric, positive-definite, block-tridiagonal linear system

\[
Ax = b,
\]

where

\[
A = \begin{pmatrix}
D_1 & A_1^T & & \\
A_2 & D_2 & A_2^T & \\
& \ddots & \ddots & \\
& \ddots & \ddots & \ddots \\
& & & D_{n-1} & A_{n-1}^T \\
& & & A_n & D_n
\end{pmatrix}
\]

The case of interest is the one for which the system (1) arises from the discretization of an elliptic partial differential equation boundary value problem and for which \( A \) is a diagonally dominant \( M \)-matrix. We restrict attention to the two-dimensional case corresponding to a standard five-point discretization with natural ordering and concomitant block structure, for which the diagonal blocks \( D_i \) are tridiagonal and strictly diagonally dominant and the off-diagonal blocks \( A_i \) have nonzero (i.e., negative) elements on the diagonal only.