THE ORDER OF B-CONVERGENCE OF ALGEBRAICALLY
STABLE RUNGE-KUTTA METHODS

K. BURRAGE and W. H. HUNSDORFER

Department of Computer Science, Centre for Mathematics and Computer Science,
University of Auckland, P.O. Box 4079,
Auckland, New Zealand 1009 AB Amsterdam, The Netherlands

Abstract.

In a previous paper it was shown that for a class of semi-linear problems many high order
Runge-Kutta methods have order of optimal B-convergence one higher than the stage order. In
this paper we show that for the more general class of nonlinear dissipative problems such a
result holds only for a small class of Runge-Kutta methods and that such methods have at most
classical order 3.

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1. Introduction.

Consider the numerical solution of a stiff initial value problem

\[ \dot{u}(t) = f(t, u(t)) \quad (t \geq 0), \quad u(0) = u_0, \]

by the Runge-Kutta method

\[ u_{n+1} = u_n + h \sum_{i=1}^{s} b_i f(t_n + c_i h, y_i), \]

\[ y_i = u_n + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, y_j) \quad (1 \leq i \leq s). \]

Here \( f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \) and \( u_0 \in \mathbb{R}^m \) are given, the real parameters \( a_{ij}, b_i, c_i \)
determine the method, \( s \) is its number of stages and \( h > 0 \) is the stepsize. The vectors \( u_n \)
approximate \( U(t) \) at \( t_n = nh \) \( (n \geq 1) \).

Let \( | \cdot | \) represent some norm on \( \mathbb{R}^m \). In this paper we will be concerned with

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bounds for the global error of the form

\[ |U(t_n) - u_n| \leq \gamma(t_n)\|U\|_{\infty}^p h^n \quad \text{for} \quad n \geq 1, \quad 0 \leq h \leq \bar{h} \]

where \( \|U\|_{\infty}^p = \max\{\|U^{(j)}(t)\| : 0 \leq t \leq t_n, \quad 1 \leq j \leq p\} \), and where \( \bar{p} \in \mathbb{N} \), \( \bar{h} > 0 \) and \( \gamma : (0, \infty) \to (0, \infty) \) are not affected by stiffness (see [13] and [16]). Let \( \mathcal{P} \) be a class of initial value problems given by (1.1). A Runge-Kutta method given by (1.2) is said to be convergent of order \( p \) on \( \mathcal{P} \) if there exist \( \bar{p} \in \mathbb{N} \), \( \bar{h} > 0 \) and \( \gamma : (0, \infty) \to (0, \infty) \) such that (1.3) holds whenever \( U \in C^{(p)}([0, \infty)) \) is a solution of a problem in \( \mathcal{P} \) and the \( u_n \) are computed from (1.2). Here it is essential that (1.3) should hold uniformly on the class \( \mathcal{P} \), not only for each problem individually. The order of convergence of method (1.2) on a given class \( \mathcal{P} \) is, by definition, the largest number \( p \) such that this method is convergent of order \( p \) on \( \mathcal{P} \).

Usually a method is said to have order \( p \) if the bound (1.3) holds individually for each problem where \( f \) is smooth and satisfies a Lipschitz condition. We will refer to this as the classical order.

In this note we consider the class of dissipative problems given by (1.1) where \( m \in \mathbb{N} \), the norm \( \cdot \) on \( \mathbb{R}^m \) is generated by an inner product \( < \cdot, \cdot > \), \( u_0 \in \mathbb{R}^m \) and \( f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m \) is a continuous function satisfying

\[ < f(t, \bar{u}) - f(t, u), \bar{u} - u > \leq 0 \quad \text{(for all} \ t \in \mathbb{R} \text{and} \ \bar{u}, u \in \mathbb{R}^m). \]

As in [13] convergence on this class of problems will be called B-convergence.

Remark 1.1. Most well-known Runge-Kutta methods satisfy \( c_i \in [0, 1] \) (for \( i = 1, 2, \ldots, s \)). For those methods which have some abscissas outside \([0, 1]\) the above definition of convergence on classes of problems should be slightly modified by taking in (1.3) \( \|U\|_{\infty}^p \) equal to \( \max\{\|U^{(j)}(t)\| : ch \leq t \leq t_{n-1} + \bar{c}h, \quad 1 \leq j \leq \bar{p}\} \), where \( \bar{c} = \min \{0, c_1, c_2, \ldots, c_s\} \) and \( \bar{c} = \max \{1, c_1, c_2, \ldots, c_s\} \). If one the \( c_i \) is negative we thus assume that the solution \( U \) of (1.1) can be extended in a smooth way on a small interval to the left of the origin.

It is well known (see [4], [9], for example) that stability of the Runge-Kutta method for all dissipative problems is guaranteed by algebraic stability

\[ BA + A^T B - bb^T \geq 0 \quad \text{and} \quad B > 0, \]

where \( A = (a_{ij}) \) and \( B = \text{diag}(b_1, b_2, \ldots, b_s) \) are \( s \times s \) matrices, \( b = (b_1, b_2, \ldots, b_s)^T \), and \( > 0 \) \((\geq 0)\) refers to positive (semi-) definiteness. Furthermore, if there exists a diagonal matrix \( D > 0 \) such that

\[ DA + A^TD > 0 \]