Abstract.

The α-type linear multistep formulas are a generalization of the Adams-type formulas. This paper is concerned with completely characterizing the $A_0$-stability of the $k$-step, order $k$ α-type formulas. Specifically, all such formulas of orders 4 or less are identified and it is shown that no α-type formulas of order 5 or more exist. These theorems generalize some previous results.

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In this paper our concern is the stability of linear multistep formulas (LMF) for the numerical solution of the ordinary differential equation

\[ y'(x) = f(x, y(x)), \quad x \in [a, b] \]
\[ y(a) = y_0 \]

where $f$ is continuous and uniformly Lipschitzian with respect to the second argument. It is assumed that the interval $[a, b]$ is partitioned with a uniform step size $h$ such that $a = x_0 < x_1 < x_2 < \ldots < x_m = b$ with $mh = b - a$. The $k$-step LMF for (1) is given by

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} \]

where $\alpha_i$ and $\beta_i$ are constants. In addition, $y_j$ and $y'_j$ are numerical approximations to $y(x_j)$ and $y'(x_j)$ for $j = 0, 1, 2, \ldots, m$.

An α-type formula is a $k$-step LMF with $\alpha_k = 1$, $\alpha_{k-1} = -\alpha$, $\alpha_{k-2} = \alpha - 1$, $\alpha_i = 0$ for $i = 0, 1, 2, \ldots, k-3$ and $0 < \alpha < 2$. The α-type formulas are important for at least two reasons. First, they reduce to the well-known Adams-type formulas when $\alpha = 1$. Second, like the Adams-type formulas, they have been

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shown to be zero-stable in a variable coefficient implementation for all order and step size changes in [9] and more generally in [10].

Our stability study is limited to the test equation \( y' = \lambda y \) where \( \lambda < 0 \). The characteristic polynomial associated with (2) on this test equation is

\[
\chi_{k,v}(\xi) = \psi_k(\xi) + v\sigma_k(\xi), \quad v = |h\lambda|
\]

where

\[
\psi_k(\xi) = \sum_{i=0}^{k} \alpha_i \xi^i, \quad \sigma_k(\xi) = \sum_{i=0}^{k} \beta_i \xi^i.
\]

Formula (2) is \( \mathcal{A}_0 \)-stable for a fixed \( k \) when the roots of \( \chi_{k,v} \) have modulus less than 1 for all \( v > 0 \).

In analyzing \( \mathcal{A}_0 \)-stability the following transformation is helpful.

\[
\zeta(z) = \frac{1+z}{1-z} \leftrightarrow z(\xi) = \frac{\xi - 1}{\xi + 1}
\]

\[
R_k(z) = \left( \frac{1-z}{2} \right)^k \psi(\zeta(z)) = \sum_{i=0}^{k} r_i z^i
\]

\[
S_k(z) = \left( \frac{1-z}{2} \right)^k \sigma(\zeta(z)) = \sum_{i=0}^{k} s_i z^i
\]

\[
X_{k,v}(z) = R_k(z) + vS_k(z) = \sum_{i=0}^{k} t_i(v) z^i.
\]

The mapping \( z \) sends \( \{\xi : \xi < 1\} \) to \( \{z : \text{Re}(z) < 0\} \) and \( \{\xi : \xi = 1\} \) to \( \{z : \text{Re}(z) = 0\} \). The \( \text{LMF} (2) \) will be \( \mathcal{A}_0 \)-stable if and only if for all \( v > 0 \), \( X_{k,v}(z) \) is a Hurwitz polynomial, i.e., a polynomial whose roots all have a negative real part.

In the determination of a Hurwitz polynomial the following result is used [1].

**Proposition.** Let \( p(z) \) be a polynomial

\[
p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots
\]

of degree \( n \neq 0 \) with real coefficients. Let \( (p(z))_1 \) be the "reduced" polynomial of degree \( n - 1 \) defined by

\[
(p(z))_1 = a_1 a_1 + (a_1 a_2 - a_0 a_3) z + a_1 a_3 z^2 + (a_1 a_4 - a_0 a_5) z^3 + \ldots
\]