CP\(^2\) as a Gravitational Instanton

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Abstract. We compare some of the properties of CP\(^2\) with those of the SU(2) Yang-Mills Instanton and conclude that CP\(^2\) may be regarded as a gravitational pseudoparticle surrounded by an event horizon.

1. Introduction

This paper is one of three concerned with Riemannian solutions of the Einstein equations with cosmological constant \(\Lambda\),

\[
R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0.
\] (1)

The first [1] contains the general theory of such spaces and their role in quantum gravity. The second (this paper) treats a particular example, CP\(^2\). The third [2] deals with generalized spin structures in Riemannian spaces, taking CP\(^2\) as a particular example.

CP\(^2\) is a two dimensional complex manifold which may also be given a Riemannian metric (known to mathematicians as the Fubini-Study metric) which satisfies (1). The fact that CP\(^2\) has non-vanishing Pontrjagin number has led Eguchi and Freund [3] to consider CP\(^2\) as an analogue of the well known "Instanton" solution of the SU(2) Yang-Mills equations [4]. What one calls an instanton outside the domain of SU(2) Yang-Mills theory depends upon which features of the Yang-Mills solutions one is making an analogy with. In this paper we shall point out some of the similarities and the differences between the two cases and relate them to the general discussion of [1]. Before doing so (in Section 6) we shall collect together some properties of CP\(^2\). Most of these are well known to mathematicians but less well known in the physics community. Section 2 contains an account of CP\(^2\) as a complex manifold, together with its standard Kähler structure. In Section 3 we discuss the isometry group (SU(3)/Z\(_3\)) and a particular 4-dimensional subgroup. The possession of a 4-dimensional isometry group acting on 3-spheres is characteristic of the Taub-NUT family of solutions of the Einstein equations and we show CP\(^2\) to be a limiting case of the general form.
We also discuss the fixed point sets and relate them to the discussion in [1]. In Section 4 we discuss the geodesics and the spectra of the basic elliptic operators defined over CP². In Section 5 we exhibit some solutions of the Maxwell and SU(2) Yang-Mills equations on this background and their connection with generalized spin structure.

Conventions. Greek indices run from 0 to 3 and latin from 1 to 3. The alternating tensor εₐₙₙₚₚₚ is \(\sqrt{g}\) if \((\alpha, \beta, \gamma, \delta) = (0, 1, 2, 3)\). The Ricci identity is

\[ V_\alpha V_\beta K^\delta = \frac{1}{2} R^\delta_{\varepsilon \alpha \beta} K^\varepsilon. \]

The Ricci tensor is \(R^\alpha_{\alpha \beta} = R^\sigma_{\sigma \alpha \beta}\). A connection on a vector bundle whose curvature \(F^\alpha_{\alpha \beta}\) is either

- self-dual: \(F^\alpha_{\alpha \beta} = \frac{1}{2} \epsilon^\alpha_{\alpha \beta \mu \nu} F^{\mu \nu} = \ast F^\alpha_{\alpha \beta}\) or
- anti self-dual: \(F^\alpha_{\alpha \beta} = -\frac{1}{2} \epsilon^\alpha_{\alpha \beta \mu \nu} F^{\mu \nu} = -\ast F^\alpha_{\alpha \beta}\)

will be called “half flat”. “Self-dual” will also be called “left flat”. In a two component SU(2) x SU(2) notation undotted indicies correspond to right handed objects. The spinor transcription of a self-dual 2-form corresponds to a symmetric 2 index undotted spinor.

2. The Manifold

\(\mathbb{C}P^2\) or complex projective two space or the projective complex plane is defined by identifying the set of triples of complex numbers \((Z_1, Z_2, Z_3)\), not all of which vanish, under the equivalence relation

\[(Z_1, Z_2, Z_3) \sim (\lambda Z_1, \lambda Z_2, \lambda Z_3), \quad \lambda \neq 0, \quad (2)\]

where \(\lambda\) is any non-zero complex number. It may be coordinatized by introducing

\[W_{ij} = Z_i / Z_j. \quad (3)\]

For fixed \(j\), provided \(Z_j \neq 0, \ W_{ij}, \ i + j, \) are a pair of complex coordinates. As \(j\) runs from 1 to 3 we obtain an atlas of 3 charts which cover \(\mathbb{C}P^2\) and are holomorphically related to one another. If

\[\zeta^1 = W_{13} = Z_1 / Z_3, \quad (4)\]
\[\zeta^2 = W_{23} = Z_2 / Z_3 \quad (5)\]

then \((\zeta^1, \zeta^2)\) cover all points for which \(Z_3 \neq 0\). This region is homeomorphic to \(\mathbb{C}^2 = \mathbb{R}^4\). The points \(Z_3 = 0\) may be regarded as “points at infinity” and are pairs \((Z_1, Z_2)\) identified under

\[(Z_1, Z_2) = (\lambda Z_1, \lambda Z_2), \quad \lambda \neq 0. \quad (6)\]

This is just \(\mathbb{C}P^1\) or \(S^2\), the familiar Riemann sphere. Thus \(\mathbb{C}P^2\) may be thought of as a compactification of \(\mathbb{R}^4\) by the addition of a sphere at infinity. Considered as a real 4-dimensional manifold \(\mathbb{C}P^2\) is compact and simply connected, with Euler number 3, Pontrjagin number 3 and second Betti number \(b_2 = 1\).