RESULTS ABOUT
MONOTONE CONVERGENCE OF
STEFFENSEN-LIKE-METHODS

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Abstract.
Certain iterations are considered which are extensions of the Steffensen method to higher dimensions. Sequences of upper and lower bounds for the solution of nonlinear equations are obtained. It is shown under which conditions the sequences converge monotonically and the convergence is quadratic.

Subject classification: 65 J, 65 H.

1. Introduction.
The Steffensen method
\[ \{ f(x_k + f(x_k)) - f(x_k) \} (x_{k+1} - x_k) / f(x_k) + f(x_k) = 0 \]
is of interest, since — under suitable conditions — it exhibits the same quadratic convergence as Newton's method, while not requiring any derivatives. This method is extended to n dimensions in [3]. For this extension there are also proved local convergence and rate of convergence results in [3].

Another extension was made by J. W. Schmidt [4] by means of the concept of a divided difference operator. Using this concept some monotone convergence results for Steffensen-like-methods are proved in [1]. In the present paper we will use a more general concept for discussing the monotone enclosure of solutions of nonlinear equations in partially ordered Banach spaces by Steffensen-like-methods.

2. Preliminaries.
The definitions and properties used in connection with the cone, which introduces a partial ordering in a Banach space B, are found in [7].

For \( x, y \in \mathbb{R}^n \), \( x \leq y \) if and only if \( x_i \leq y_i \), \( i = 1(1)n \). \( S(B) \) means the set of all continuous, linear operators \( (B \rightarrow B) \).

Kantorovich's Lemma ([2]): Let \( B \) be a Banach space, which is partially ordered by a closed, regular cone and \( A([x,y] \subset B \rightarrow B) \) a continuous, isotone mapping. If \( x \leq A(x) \) and \( A(y) \leq y \), then \( A \) has a fix-point in \([x, y] \).

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In this paper \( B \) denotes a Banach space, which is partially ordered by a closed regular cone. In order to enclose solutions of \( F(x) = 0, \quad F(D \subseteq B \rightarrow B) \), we consider the following iteration methods

\[
\begin{align*}
F(x_k) + A(x_k, x_k + QF(x_k)(x_{k+1} - x_k) = 0 \\
F(y_k) + A(x_{k+1}, x_k + QF(x_{k+1})(y_{k+1} - y_k) = 0
\end{align*}
\]

\( k = 1, 2, \ldots \)

(3.1)

\[
\begin{align*}
F(x_k) + A(x_k, x_k + QF(x_k)(x_{k+1} - x_k) = 0 \\
F(y_k) + A(x_k, x_k + QF(x_k)(y_{k+1} - y_k) = 0
\end{align*}
\]

\( k = 1, 2, \ldots \)

(3.2)

\( A \) denotes a mapping \((M = \{(u, v) \mid x_1 \leq u \leq v \leq y_1\} \rightarrow S(B))\) which satisfies

\[
F(x) - F(y) \leq A(x, y)(x - y), \quad (x, y) \in M.
\]

(3.3)

Using a specific operator \( Q(B \rightarrow B) \) the following theorem can be proved:

**Theorem 3.1.** Let \( F \) be a continuous mapping \(([x_1, y_1] \subseteq B \rightarrow B)\) such that \( F(x_1) \geq 0 \geq F(y_1) \). Assume that \( A(M \rightarrow S(B)) \) is a mapping with the properties (3.3) and

\[
A(u_1, v_1) \geq A(u_2, v_2), \quad u_1 \geq u_2, \quad v_1 \geq v_2.
\]

(3.4)

If there exists a nonnegative injective mapping \( T \in S(B) \) and a mapping \( G(B \rightarrow B) \) such that

\[
TG \leq I, \quad 0 \leq T(G + A(u, v)),
\]

(3.5)

and an operator \( Q(B \rightarrow B) \) with the property

\[
0 \leq Q \leq T,
\]

(3.6)

then the iterations (3.1) and (3.2) are well defined. This means there exist solutions \( x_{k+1}, y_{k+1} \) of the linear equations (3.1) and (3.2). The monotone sequences \( (x_k), (y_k) \) have limits \( x^*, y^* \), which are solutions of \( F(x) = 0 \), and the monotone enclosure

\[
x_1 \leq \ldots \leq x_k \leq x_{k+1} \leq \ldots \leq x^* \leq y^* \leq \ldots \leq y_{k+1} \leq y_k \leq \ldots \leq y_1
\]

(3.7)

holds for any solution \( z^* \in [x_1, y_1] \) of \( F(x) = 0 \).

If we assume that there exists a mapping \( S(B \rightarrow B) \) with the properties

\[
F(z_1) - F(z_2) \leq -S(z_1 - z_2), \quad z_1 \geq z_2, \quad 0 \leq S^{-1},
\]

(3.8)

then \( x^* = \lim x_k = \lim y_k = y^* \).

If there exists an operator \( \bar{S}(B \rightarrow B) \) with the properties

\[
\bar{S} \leq -A(y_1, y_1), \quad 0 \leq \bar{S}^{-1} \in S(B), \quad \text{and we assume}
\]

(3.9)

\[
\|F(z_1) - F(z_2) - A(z_2, z_2)(z_1 - z_2)\| \leq \beta \|z_1 - z_2\|^2, \quad z_1 \geq z_2
\]

(3.10)

then the R-order of the iterations (3.1) and (3.2) at \( x^* \) is not less than 2.