The Group with Grassmann Structure $UOSP(1.2)$

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Abstract. The finite-dimensional representations of the Lie superalgebra $osp(1.2)$ and the group with Grassmann structure $OSP(1.2)$ have been studied. The explicit expression of the projection operator of the superalgebra $osp(1.2)$ has been found. The operator permits an arbitrary finite-dimensional representation to be expanded in the components multiple to the irreducible ones. The Clebsch-Gordan coefficients for the tensor product of two arbitrary irreducible representations have been obtained. The matrix elements of the irreducible representations of the group $UOSP(1.2)$ [the analogue of the compact form of the group $OSP(1.2)$] are studied. The explicit form of these matrix elements, the differential equations satisfied by them, and the integral of their product have been found.

1. Introduction

The Lie superalgebras and the Lie groups with Grassmann structure$^1$ have been extensively used since recently in physics. These objects appeared first in the problems relevant to the secondary quantization of the fermion systems [5], then in the dual model, and finally in the supergravity and the supersymmetric field theory (see the review in [6]). The natural problem arises, therefore, to develop a formalism of the theory of representation of the Lie superalgebras and groups with Grassmann structure up to the extent as was achieved for some of the semisimple Lie groups [14].

The present work studies in detail the finite-dimensional representations of the Lie superalgebra $osp(1.2)$ and the generated group with Grassmann structure $OSP(1.2)$. The representations of the Lie superalgebra $osp(1.2)$ were studied earlier in [8, 10, 11].

In the first part of the present work, the projection operator method developed earlier for the usual semisimple Lie algebras [1, 2] is applied to the superalgebra $osp(1.2)$.
osp(1.2). This technique has been used to obtain the explicit expressions of the Clebsch-Gordan coefficients for the tensor product of the arbitrary irreducible representations of the superalgebra \(osp(1.2)\).

Studied in the second part are the matrix elements of the irreducible representations of the group \(UOSP(1.2)\), i.e. the analogue of the compact form of the group \(OSP(1.2)\). We shall find the explicit form of these matrix elements, the differential equations satisfied by them, and the integrals of their product.

2. The Basis of Finite-Dimensional Representation with the Highest Weight of the Superalgebra \(osp(1.2)\)

Let some of the known definitions be reminded \([7]\). A complex (or real) linear space \(V\) is called \(Z_2\)-graded if it is presented in the form of the direct sum of two subspaces, i.e. \(V = V_0 \oplus V_1\). The elements \(V_0\) are called even, and those of \(V_1\) odd. The elements that are either even or odd are called homogeneous. There exists a parity function \(\varepsilon\) defined on homogeneous elements by the formula:

\[
\varepsilon(x) = \begin{cases} 
0, & \text{if } x \in V_0 \\
1, & \text{if } x \in V_1.
\end{cases}
\]  

(2.1)

The Lie superalgebra (or, which is the same, the \(Z_2\)-graded Lie algebra) is the \(Z_2\)-graded space \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) with bilinear operation (called commutator) satisfying the following axioms

\[
[x, y] = (-1)^{\varepsilon(x)\varepsilon(y) + 1}[y, x],
\]

\[
(-1)^{\varepsilon(z)\varepsilon(x)}[x, [y, z]] + (-1)^{\varepsilon(z)\varepsilon(y)}[z, [x, y]] + (-1)^{\varepsilon(y)\varepsilon(x)}[y, [z, x]] = 0.
\]  

(2.2)

for all the homogeneous elements \(x, y, z\). The commutator \([x, y]\) will also be designated \([x, y]\) if \(\varepsilon(x) = 0\) or \(\varepsilon(y) = 0\) and \([x, y]\) if \(\varepsilon(x) = \varepsilon(y) = 1\).

A representation of a superalgebra \(\mathfrak{g}\) in a finite-dimensional graded vector space \(V = V_0 \oplus V_1\) is the realization of the algebra \(\mathfrak{g}\) by the operators \(T_x\) in \(V\) subject to the condition: if \(x \in \mathfrak{g}_0\) then \(T_x V_0 \subseteq V_0\), \(T_x V_1 \subseteq V_1\) and if \(x \in \mathfrak{g}_1\) then \(T_x V_0 \subseteq V_1\), \(T_x V_1 \subseteq V_0\). We assume that there exists on \(V\) such a nondegenerate bilinear Hermitian form (denoted by brackets \((|)\)) that \(V_0\) and \(V_1\) are orthogonal with respect to this form, i.e.,

\[
\langle V_0 | V_1 \rangle = 0.
\]  

(2.3)

The elements \(L_\pm, L_0, R_\pm\) satisfying the conditions

\[
[L_0, L_\pm]_\pm = \pm L_\pm, \hspace{1cm} [L_+, L_-]_\pm = 2L_0,
\]  

(2.4a)

\[
[L_0, R_\pm]_\pm = \pm \frac{1}{2} R_\pm, \hspace{1cm} [L_\mp, R_\pm]_\mp = R_\pm, \hspace{1cm} [L_\pm, R_\pm]_\pm = 0,
\]  

(2.4b)

\[
[R_\pm, R_\pm]_\mp = \pm \frac{1}{2} L_\pm, \hspace{1cm} [R_\mp, R_\mp]_\mp = -\frac{1}{2} L_0
\]  

(2.4c)

form the Cartan-Weyl basis of the superalgebra \(osp(1.2)\) \([8]\). The elements \(L_\pm, L_0\) form the basis in the even subspace \(osp_0(1.2)\)\(^2\), while \(R_\pm\) form the basis in the odd subspace \(osp_1(1.2)\):

\[
osp(1.2) = osp_0(1.2) \oplus osp_1(1.2).
\]  

(2.5)

\(^2osp_0(1.2)\) is the simple Lie algebra \(A_1\)