The Local Structure of the Spectrum of the One-Dimensional Schrödinger Operator

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Abstract. Let \( H_V = -\frac{d^2}{dt^2} + q(t, \omega) \) be an one-dimensional random Schrödinger operator in \( \mathbb{L}^2(-V, V) \) with the classical boundary conditions. The random potential \( q(t, \omega) \) has a form \( q(t, \omega) = F(x_t) \), where \( x_t \) is a Brownian motion on the compact Riemannian manifold \( K \) and \( F : K \rightarrow \mathbb{R}^1 \) is a smooth Morse function, \( \min F = 0 \). Let \( N_V(\Delta) = \sum_{E_i(V) \in \Delta} 1 \), where \( \Delta \in (0, \infty) \), \( E_i(V) \) are the eigenvalues of \( H_V \). The main result (Theorem 1) of this paper is the following. If \( V \rightarrow \infty \), \( E_0 > 0 \), \( k \in \mathbb{Z}_+ \) and \( a > 0 \) (\( a \) is a fixed constant) then

\[
\lim_{V \rightarrow \infty} \mathbb{P}\left\{ N_V\left( E_0 - \frac{a}{2V}, E_0 + \frac{a}{2V} \right) = k \right\} = e^{-anE_0}(anE_0)^k k!,
\]

where \( n(E_0) \) is a limit state density of \( H_V \), \( V \rightarrow \infty \). This theorem mean that there is no repulsion between energy levels of the operator \( H_V \), \( V \rightarrow \infty \).

The second result (Theorem 2) describes the phenomenon of the repulsion of the corresponding wave functions.

1.

In a series of latest works in physics (see [1]) the phenomenon of the repulsion of the energy levels in the spectrum of complicated (random) quantum systems was discussed. The formal definitions are the following.

Let \( H_V \) be the family of the Hamiltonians describing the behaviour of the system in the volume \( V \) and let \( E_1^{(V)} < E_2^{(V)} \leq \ldots \) be the corresponding energy levels. In various interesting cases these levels are thickening in the limit and moreover for every \( \alpha > 0 \) \( E_2^{(V)} - E_1^{(V)} \rightarrow \alpha \) as \( |V| \rightarrow \infty \).

We shall consider two neighbour levels \( E_n^{(V)} \) and \( E_{n+1}^{(V)} \), where \( n \sim \alpha |V| \). It is natural to suppose that the normalized "spectral split" \( \Delta_n = (E_{n+1} - E_n)/M(E_{n+1} - E_n) \) has a limit distribution as \( |V| \rightarrow \infty \), i.e. there exists

\[
\lim_{|V| \rightarrow \infty} \mathbb{P}\{\Delta_n < x\} = G(x).
\]
If \( G(x) = o(x) \) when \( x \to 0 \) in this case we deal with the repulsion of the levels (near \( E_\alpha \)); if \( G(x) \sim cx \) we say that an interaction of the levels does not exist near \( E_\alpha \); in the case \( G(x)/x \xrightarrow{x \to 0} \infty \) we may say that there is an attraction between levels (or that the levels show a tendency to group).

It is natural to study several levels near \( E_\alpha \). Mathematically this problem is reduced to the analysis of the joint limit distribution of the several neighbour spectral splits \( A_n, A_{n+1}, \ldots, A_{n+k-1} \).

As far as it is known to the author no rigorous result in this field has yet been published. However in the so called Wigner Gaussian symmetrical ensemble \( H_n = (\xi_{ij}), i,j = 1, 2, \ldots, n \) (\( \xi_{ij}, i > j \) are independent Gaussian random values) the limit distribution function averaged in all splits was found, i.e.

\[
\tilde{G}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P\{A_k < x\}
\]

and the existence of repulsion was established [2–4].

For the unordered structures the spectrum of which coincides with that of the Schrödinger operator with the random potential the repulsion of the levels was also asserted in [5]. A number of physics works following [5] were based on the results of [5], but it turned out that [5] was false. It is possible to prove the absence of the interaction between the levels in unordered one-dimensional structures for a large class of random stationary potentials, in particular, for \( \delta \)-potential explored in [5]. Moreover it is possible to analyse the local structure of the spectrum near the fixed point \( E_\alpha \) in full. This spectrum proves to be a Poisson flow near \( E_\alpha \) on the natural scale, i.e. the neighbour spectral splits (asymptotically as \( |V| \to \infty \)) are independent and have exponential distribution.

Our paper contains the proof of the above formulated results and is close to [6, 7].

For the sake of convenience of the references to [7] we narrow the class of the studied potentials but our results remain true for the Kronig-Penny potential and for the potential of the “white noise” type.

2.

Now we pass on to exact formulations. We consider the Schrödinger operator of the Markov type which has been introduced in [6, 7], namely

\[
H = -\frac{d^2}{dt^2} + F(x_t(\omega)), \quad t \in \mathbb{R}^1, \quad \omega \in \Omega.
\]  (1)

Here \( \Omega \) is the probability space with the measure \( \mathbf{P} \) (this space may be identified with the ensemble of all the realizations of the process \( x_t, t \in \mathbb{R}^1 \), \( x_t(\omega) \) is the Brownian motion on the compact Riemannian manifold \( K \) and has the generating operator \( \Delta \)). The invariant measure of \( x_t \) is the natural Riemannian measure. Taking \( dx \) to be the initial distribution we turn \( x_t \) into stationary Markov process with “good” mixing properties. The function \( F: K \to \mathbb{R}^1 \) is smooth (\( C^\alpha \)) and “non-flat” (see [6, 7]). The last is fulfilled when \( F \) has a finite number of nondegenerate (Morse) critical points.