The Local Structure of the Spectrum of the One-Dimensional Schrödinger Operator

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Abstract. Let $H_{V} = -\frac{d^{2}}{dt^{2}} + q(t, \omega)$ be an one-dimensional random Schrödinger operator in $\mathcal{L}^{2}(-V, V)$ with the classical boundary conditions. The random potential $q(t, \omega)$ has a form $q(t, \omega) = F(x_{i})$, where $x_{i}$ is a Brownian motion on the compact Riemannian manifold $K$ and $F : K \to \mathbb{R}$ is a smooth Morse function, $\min F = 0$. Let $N_{V}(\lambda) = \sum_{\lambda \in \mathcal{E}_{V}} 1$, where $\mathcal{E} \subset (0, \infty)$, $\mathcal{E}_{V}$ are the eigenvalues of $H_{V}$. The main result (Theorem 1) of this paper is the following. If $V \to \infty$, $E_{0} > 0$, $k \in \mathbb{Z}$, and $a > 0$ ($a$ is a fixed constant) then

$$\lim_{V \to \infty} n(E_{0}) = k$$

where $n(E_{0})$ is a limit state density of $H_{V}$, $V \to \infty$. This theorem mean that there is no repulsion between energy levels of the operator $H_{V}$, $V \to \infty$.

The second result (Theorem 2) describes the phenomenon of the repulsion of the corresponding wave functions.

In a series of latest works in physics (see [1]) the phenomenon of the repulsion of the energy levels in the spectrum of complicated (random) quantum systems was discussed. The formal definitions are the following.

Let $H_{V}$ be the family of the Hamiltonians describing the behaviour of the system in the volume $V$ and let $E_{1}(V) < E_{2}(V) \leq \ldots$ be the corresponding energy levels. In various interesting cases these levels are thickening in the limit and moreover for every $\varepsilon > 0$ $E_{n+1}(V) - E_{n}(V)$ as $|V| \to \infty$.

We shall consider two neighbour levels $E_{n}(V)$ and $E_{n+1}(V)$, where $n \sim \varepsilon|V|$. It is natural to suppose that the normalized "spectral split" $A_{n} = (E_{n+1} - E_{n})/M(E_{n+1} - E_{n})$ has a limit distribution as $|V| \to \infty$, i.e. there exists

$$\lim_{|V| \to \infty} P\{A_{n} < x\} = G(x).$$
If $G(x) = o(x)$ when $x \to 0$ in this case we deal with the repulsion of the levels (near $E_{\alpha}$); if $G(x) \sim cx$ we say that an interaction of the levels does not exist near $E_{\alpha}$; in the case $G(x)/x \xrightarrow[x \to 0]{} \infty$ we may say that there is an attraction between levels (or that the levels show a tendency to group).

It is natural to study several levels near $E_{\alpha}$. Mathematically this problem is reduced to the analysis of the joint limit distribution of the several neighbour spectral splits $A_n, A_{n+1}, \ldots, A_{n+k-1}$.

As far as it is known to the author no rigorous result in this field has yet been published. However in the so called Wigner Gaussian symmetrical ensemble $H_n = (\xi_{ij}), i,j=1,2,\ldots,n$ ($\xi_{ij}, i \geq j$ are an independent Gaussian random values) the limit distribution function averaged in all splits was found, i.e.

$$\bar{G}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P\{A_k < x\}$$

and the existence of repulsion was established [2–4].

For the unordered structures the spectrum of which coincides with that of the Schrödinger operator with the random potential the repulsion of the levels was also asserted in [5]. A number of physics works following [5] were based on the results of [5], but it turned out that [5] was false. It is possible to prove the absence of the interaction between the levels in unordered one-dimensional structures for a large class of random stationary potentials, in particular, for $\delta$-potential explored in [5]. Moreover it is possible to analyse the local structure of the spectrum near the fixed point $E_{\alpha}$ in full. This spectrum proves to be a Poisson flow near $E_{\alpha}$ on the natural scale, i.e. the neighbour spectral splits (asymptotically as $|V| \to \infty$) are independent and have exponential distribution.

Our paper contains the proof of the above formulated results and is close to [6, 7].

For the sake of convenience of the references to [7] we narrow the class of the studied potentials but our results remain true for the Kronig-Penny potential and for the potential of the "white noise" type.

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Now we pass on to exact formulations. We consider the Schrödinger operator of the Markov type which has been introduced in [6, 7], namely

$$H = -\frac{d^2}{dt^2} + F(x_t(\omega)), \quad t \in R^1, \quad \omega \in \Omega.$$  \hspace{1cm} (1)

Here $\Omega$ is the probability space with the measure $\mathbf{P}$ (this space may be identified with the ensemble of all the realizations of the process $x_t, t \in R^1, x_t(\omega)$ is the Brownian motion on the compact Riemannian manifold $K$ and has the generating operator $A$). The invariant measure of $x_t$ is the natural Riemannian measure. Taking $dx$ to be the initial distribution we turn $x_t$ into stationary Markov process with "good" mixing properties. The function $F : K \to R^1$ is smooth ($C^\infty$) and "non-flat" (see [6, 7]). The last is fulfilled when $F$ has a finite number of nondegenerate (Morse) critical points.