THE NORMAL CURVATURES OF A GRAPH

A set of arithmetical invariants for each vertex of a graph was defined in [1]. In this paper these invariants are expressed by the eigenvalues and eigenvectors of the graph.

With a simple graph whose adjacency matrix is \( G = \{g_{ik}\} \) we can associate the surface

\[ 2z = x^T \cdot G \cdot x \]  

(1)

in the \((n+1)\)-dimension Euclidean space. The vertex \( v_i \) corresponds to the point

\[ M_i : z = 0, \quad x_1 = 1, \quad x_j = 0 \quad (j \neq i) \]  

(2)

on the surface (1). The normal curvatures of the surface form a set of Euclidean invariants for the point \( M_i \) and, consequently, a set of arithmetical invariants for each vertex \( v_i \).

The most important part of the equation, that determines the normal curvatures, is the polynomial [1]

\[ \Delta_i(k) = \chi(k) + k \cdot \chi_i(k), \]  

(3)

where \( \chi(k) \) and \( \chi_i(k) \) are, respectively, the characteristic polynomials of \( G \) and \( G_i = G - v_i \). The coefficients of \( \Delta_i(k) \)
form a set of invariants for each vertex \( v_i \). The leading coefficient is the valency of \( v_i \), the second is twice the number of triangles through \( v_i \) and so on. The absolute term is \( \text{det} \ G \) for each \( v_i \).

Given the set of subgraphs \( G_i \) we can determine the polynomial

\[
\Delta_i(k) = \Delta_i(k) - \text{det}(G)
\]

Indeed, since \( \frac{dX(k)}{dk} = \sum_i \chi_i(k) \) the calculation is obvious. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of \( G \) and the elements of the \( k \)-th row of \( C = [x_i^k] \) are the coordinates of the eigenvector \( \hat{e}_k \) which corresponds to \( \lambda_k \).

In the system of coordinates, formed by \( \hat{e}_k \), the matrix \( G \) is diagonal and the \( i \)-th column of \( C \) contains the coordinates of \( M_i \).

By calculating the quadratic forms of (1) in this system of coordinates, we obtain

\[
\Delta_i(k) = (-k)^{n-2}\{\lambda_1(x_i^1)^2(\lambda_2 + \lambda_3 + \ldots + \lambda_n) + \\
+ \lambda_2(x_i^2)^2(\lambda_1 + \lambda_3 + \ldots + \lambda_n) + \ldots + \\
+ \lambda_n(x_i^n)^2(\lambda_1 + \lambda_2 + \ldots + \lambda_{n-1})\} + \\
+ (-k)^{n-3}\{\lambda_1(x_i^1)^2(\lambda_2\lambda_3 + \lambda_2\lambda_4 + \ldots + \lambda_{n-1}\lambda_n) + \\
+ \lambda_2(x_i^2)^2(\lambda_1\lambda_3 + \lambda_1\lambda_4 + \ldots + \lambda_{n-1}\lambda_n) + \\
+ \lambda_n(x_i^n)^2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_{n-2}\lambda_{n-1})\} + \\
\ldots + \lambda_1\lambda_2 \ldots \lambda_n
\]

Formula (5) shows that for two vertices \( v_i \) and \( v_m \) with

\[
z^j = (x_i^j)^2 - (x_m^j)^2 = 0 \quad (j = 1, 2, \ldots, 4)
\]

we have