ON BLOCKING SETS IN SYMMETRIC BIBD's WITH \( \lambda \geq 2 \).

Marialuisa J. de Resmini

Blocking sets in symmetric BIBD's with \( \lambda \geq 2 \) are defined and bounds for their sizes are found. Thus, some results known in projective planes are generalized to symmetric BIBD's.

Recall that a BIBD with parameters \((v,b,r,k,\lambda)\) is a pair \((\mathcal{S},\mathcal{B})\), where \(\mathcal{S}\) is a \(v\)-set - whose elements will be called points - and \(\mathcal{B}\) is a collection of \(k\)-subsets of \(\mathcal{S}\) - blocks - such that each point is contained in \(r\) blocks and through any two points there are \(\lambda\) blocks.

When \(b = v\), the BIBD is symmetric (SBIBD); then \(r = k\) and any two blocks meet in \(\lambda\) points.

Let \(B = (\mathcal{S},\mathcal{B})\) be a \((v,k,\lambda)\) SBIBD. A subset \(S\) of \(\mathcal{S}\) will be called a blocking set in \(B\) if the following hold:

(i) each block of \(B\) meets \(S\) in at least one point;

(ii) \(S\) contains no block.

Then, the complement \(\mathcal{S} \setminus S\) is again a blocking set.

A blocking set will be called irreducible if, for any \(x\) in \(S\), \(S \setminus \{x\}\) is not a blocking set.

Let \(S\) be an irreducible blocking set in a projective plane of order \(n\). In [1], [3] it was proved that

\[
(1) \quad n + \sqrt{n} + 1 \leq |S| \leq n^2 - \sqrt{n} ;
\]

i.e., in a projective plane, the size of a Baer subplane is a lower bound
for the size of a blocking set.

In [4], [5] it was proved that a type (1,n) h-set in a SBIBD B(v,k,\(\lambda\)) (i.e. a subset of the point set met by any block either in 1 or in n points) is either a Baer subdesign or a Hermitian subset, and

\[ n = 1 + \sqrt{k - \lambda} \]

Obviously, both a Baer subdesign and a Hermitian subset are blocking sets. On the other hand, any \((k - \lambda + \alpha)\)-subset of a block, \(1 \leq \alpha \leq \lambda\), is a blocking set in \(B(v,k,\lambda)\).

The main result will now be proved.

Theorem. - If \(S\) is an irreducible blocking set in a SBIBD \(B(v,k,\lambda) = B\), then

\[ \frac{k - \lambda}{\lambda} \leq |S| \leq \frac{k^2 - 2k + \lambda - \sqrt{k - \lambda}}{\lambda} \]

Proof. Firstly, remark that the lower bound in (2) is the size of a Baer subdesign and the upper bound is the size of its complement.

In order to prove (2), we start proving that

\[ \lambda h > k - 1, \]

where \(h = |S|\).

Since a blocking set is a set of class \([1,2,\ldots,k - 1]\), the following hold [4], [5]:

\[
\begin{align*}
\sum_{i=1}^{k-1} t_i &= v \\
\sum_{i=1}^{k-1} i t_i &= kh \\
\sum_{i=2}^{k-1} i(i - 1) t_i &= \lambda h(h - 1),
\end{align*}
\]

\(t_i\) being the number of \(i\)-secant blocks.

Subtracting the first two equations (4), we get

\[ \sum_{i=2}^{k-1} (i - 1) t_i = kh - v > 0. \]