A PASCAL THEOREM APPLIED TO MINKOWSKI GEOMETRY

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If on an oval in a projective plane a 4-point Pascal theorem, $\pi$, with fixed points $U$ and $V$ holds, then the oval is $\{(x,y)|xy=c\} \cup \{(0)\cup \{\infty\}$, with $c \neq 0$, in some Hall coordinatization. If for every 3 distinct points $P$, $Q$, $R$ (not on $UV$; neither $U$ nor $V$ collinear with two of $P$, $Q$, $R$) there is through them a certain point set satisfying an extended version of $\pi$, then all these sets together with all lines not through $U$ or $V$ form the circles of a plane Minkowski (= pseudo-euclidean) geometry over a commutative field. $\pi$ may be expressed in terms of Minkowski geometry. Together with incidence axioms derived from the projective incidence axioms, the Minkowski version of $\pi$ characterizes the plane Minkowski geometry over a commutative field and is thus equivalent to Miquel's theorem.

The validity of Pascal's theorem on an oval in a projective plane has been shown to make the plane pappian and the oval a conic [2, 4, 9]. Special Pascal theorems have been studied: Hofmann [5] showed that the 5-point specialization is equivalent to the full 6-point theorem, and other cases were discussed in [7]. There are two 4-point specializations one of which, here called $\pi_{UV}$, has the property that its validity with two fixed points $U$ and $V$ on the oval "makes the oval a conic", in the sense that a coordinate system can then be found in which the proper points of the oval are $\{(x,y)|xy=c\}$ for some $c \neq 0$. This property motivated a search for a role played by $\pi_{UV}$ in the foundations of plane Minkowski (= pseudo-euclidean)
geometry whose classical affine model [cf. 3] is based on the hyperbolas $xy=c$ and their translates. Indeed it turns out (Theorem 2) that a projective plane in which $\pi_{UV}$ holds for certain point sets provides a model for a plane Minkowski geometry over a commutative field. When we free ourselves from the projective model, then $\pi_{UV}$ can now be expressed in a purely Minkowski-geometric form, called $\pi_X$. The axioms of the projective plane used for constructing the model can also be translated into Minkowski-geometric versions. These new axioms, together with $\pi_X$, now provide a foundation for the plane Minkowski geometry over a commutative field. This foundation seems to be less redundant than previously used axiom systems [3, 6]. Another interesting consequence is that $\pi_X$ is equivalent to Miquel's theorem.

1. In a projective plane.

The following special 4-point Pascal theorem on an oval will be called $\pi$.

$\pi$: If $U$, $V$, $P$, $Q$ are distinct points of the oval, and if the tangents at $U$ and $V$ are called $UU$ and $VV$, respectively, then the points $UU \cap VV$, $UP \cap VQ$, and $UQ \cap VP$ are collinear.

In particular, we will denote the theorem $\pi$ for fixed points $U$ and $V$ by $\pi_{UV}$.

**THEOREM 1.** An oval in a projective plane can be represented by $\{(x,y)|xy=c \neq 0\} \cup U \cup V$ in some Hall coordinate system [cf. 1, p. 203] if $\pi_{UV}$ holds true on the oval.

A special case of the theorem can be found in [7].

**Proof.** In a Hall coordinate system let $U = (0)$, $V = (\infty)$, and $P = (1,c)$ a distinct arbitrary point of the oval. For every point $Q = (x,y)$ of the oval, distinct from $P$, $U$, $V$, we have $UP \cap VQ = (x,c)$ and $UQ \cap VP = (1,y)$. 

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