DIRECT SUMMANDS OF VECTOR GROUPS

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In Theorem 4.3 in [3] we stated the following: If \( V = A \oplus B \) is a reduced vector group and \( |V| \) is non-measurable, then \( A \) and \( B \) are vector groups. Our proof in [3], however, was defective in part. We here present a new proof which is also much simpler than that in [3]. Explicitly we prove the following.

**Theorem.** Let \( V = \Pi_i R_i = A \oplus B \) where each \( R_i \) is a rank-one reduced torsion-free abelian group and \( |I| \) is non-measurable. Then \( A \) (also \( B \)) is a direct product of rank-one groups.

**Preliminaries.** We shall presume a basic knowledge of vector groups and slender groups such as is found in [2] (see also Lemmas 3.1 and 3.2 in [3]). The maps \( \alpha: V \to A \), \( \beta: V \to B \), \( \alpha_i: V \to R_i \) will be the obvious projections. The type of \( R_i \) is denoted \( t_i \). The first infinite ordinal is \( \omega \). We first give 4 lemmas and then the formal proof.

**Lemma 1.** \( I \) can be ordered as an ordinal so that, for each limit ordinal \( j \), \( \alpha_i(R_k) = 0 \) whenever \( i < j \leq k \).

**Proof.** Let \( 0 \in I \) be arbitrary. Suppose \( k \) is an ordinal and ordinals \( i \) have been chosen from \( I \) for all \( i < k \). Choose \( k \) from \( I \setminus \{ i: i < k \} \) so that: \( \alpha_i(R_k) \neq 0 \) for minimal \( i < k \) if possible; otherwise choose \( k \) arbitrarily.

Inductively totally order \( I \) in this manner. Since each \( R_i \) is slender, it is easy to see that \( I \) will satisfy the lemma for \( j = \omega \) and eventually for all limit ordinals \( j \).

**Lemma 2.** Suppose \( J \) is a well-ordered set with least member \( 0 \) and \( A \) has direct summands \( A_j \), \( A^j \) for each \( j \) in \( J \) such that:

1. \( A = A^0 \) and \( A^j = A_j \oplus A^{j+1} \) (if \( j \) is maximal \( A^j = 0 \)),
2. \( A^k = \bigcap_{j < k} A^j \) if \( k \) is a limit of members of \( J \),
3. for each \( i \in I \) \( \alpha(A_j) = 0 \) for almost all \( j \).
4. \( \bigcap_j A^j = 0 \).

Then \( A = \prod_j A_j \).

**Proof.** For simplicity assume \( J \) is an ordinal. a) By (3) \( \sum_j A_j \subseteq A \) (see Lemma 3.2 in [3]). b) Suppose \( a \in A \) and we have chosen \( a_j \in A_j \) for all \( j < k \) such that \( a - \sum_{j < k} a_j \in A^k \). By repeated use of (1) we can find \( a_j \in A_j \) for all \( j \) where \( j - k \)}
is finite so that $a_i = \sum_{j=k+i}^{j=k+1} a_j \in A^{k+1}$ for all finite $i$. Now $a_i = \sum_{j=k+i}^{j=k+1} a_j \in A^{k+1}$ by (2). We can now proceed inductively to find $a_j \in A_j$ for all $j$ so that $a_i = \sum_{j=k+i}^{j=k+1} a_j \in A^k$ for each $k$ in $J$. So, for any $k$, $a_i = \sum_{j=k+i}^{j=k+1} a_j = (a_i - \sum_{j=k+1}^{j=k+1} a_j) + \sum_{j=k+1}^{j=k+1} a_j$, which is in $A^k$ since the left sum is in $A^k$ by construction and the right sum is in $A^k$ by (3) and Lemma 3.2 in [3]. Therefore $a_i = \sum_{j=k+i}^{j=k+1} a_j = 0$ by (4) and $A \subseteq \sum_{j=k}^{j=k+1} A_j$. c) Suppose $\sum_{j=k}^{j=k+1} a_j = 0$ and $a_k \neq 0$ for minimal $k$. Then $0 \neq a_k = -\sum_{j=k+i}^{j=k+1} a_j \in A^{k+1} \cap A_k = 0$, a contradiction. So $a_j = 0$ for all $j$. By a), b), c) $A = \prod_{j=k}^{j=k+1} A_j$.

Lemmi 3. Let $I$ be ordered as in Lemma 1 and let $J$ consist of 0 and all limit orderinals in $I$. For each $j$ in $J$ write: $V_j = \sum_{i,j} R_i$, $V_j = \prod_{i,j} R_i$, $A_j = A \cap (\beta(V_j) \oplus V_j)$ and $B_j = B \cap (\alpha(V_j) \oplus V_j)$ (if $j$ is maximal in $J$ set $V_j$ = 0). Then $A = \prod_{j=k}^{j=k+1} A_j$ and $V_j = A_j \oplus B_j$ for each $j$.

Proof. For each $j$ $\alpha(V_j) \subseteq V_j$ by Lemma 1 so $V_j = \alpha(V_j) \oplus \beta(V_j)$. Let $A^j = \alpha(V_j)$ and $B^j = \beta(V_j)$. Now $V_j = A^j \oplus B^j$, $V_j = A^j \oplus B^j$ and $V_j = A^j \oplus B^j \oplus B^j$. Consequently $A^j = A_j \oplus A^j \oplus B^j$ and $V_j = A^j \oplus B^j \oplus B^j$ as desired. We now just apply Lemma 2. Clearly (1) is true. If $k$ is a limit of members of $J$ then $A^k \subseteq \bigcap_{j<k} A^j \subseteq \bigcap_{j<k} V_j = \alpha(V^k) = A^k$ so (2) is true. For fixed $i$ consider $\alpha_i(\prod_{j=k}^{j=k+1} V_j)$. By the slenderness of $R_i$, $\alpha_i(V_j) = 0$ for almost all $j$. But $\alpha(A_j) = A_j$ and $A_j$ by definition is in $\alpha(V_j)$. So $\alpha_i(A_j) \subseteq \alpha_i(V_j) = 0$ for almost all $j$ and (3) is true. Finally $\bigcap_{j<k} A^j \subseteq V_j = 0$ (if $j$ is maximal in $J$ include $j+\omega$ here). Lemma 2 completes the proof.

Lemma 4. Suppose $m \in I$ and $A = C \oplus D$ where $C$ has finite rank. Then $D = E \oplus F$ where $E$ has finite rank and $\alpha_m(F) = 0$.

Proof. Write $V^m = \prod_{i=m}^{i=m} R_i$ and, for each type $t$, let $V_t = \prod_{i=m}^{i=m} R_i$. Let $\gamma : A \rightarrow C$ be the projection. Consider $\alpha_m(\Pi V_t)$ and $\gamma(\Pi V_t)$ where $V = \Pi V_t$. Since $R_m$ and $C$ are slender, $\alpha_m(V_t) = 0 = \gamma(V_t)$ for almost all $t$. So $\alpha_m(V_t) \subseteq D \cap V^m$ for almost all $t$. Let $T$ be the finite set of types $t$ such that $\alpha_m(V_t) \subseteq D \cap V^m$. We will induct on the order of $T$ to prove the lemma. If $T = \emptyset$, then $A = \alpha(V_t) \subseteq D \cap V^m$ and $E = 0$, $F = D$ satisfy the lemma. If $T \neq \emptyset$, let $s$ be a minimal type in $T$. Set $T^* = T - s$ and $A = \alpha(V_t) \subseteq D \cap V^m$ and $E = 0$, $F = D$ satisfy the lemma. If $T \neq \emptyset$, let $s$ be a minimal type in $T$. Set $T^* = T - s$ and $A = \alpha(V_t) \subseteq D \cap V^m$ and $E = 0$, $F = D$ satisfy the lemma. If $T \neq \emptyset$, let $s$ be a minimal type in $T$. Set $T^* = T - s$ and $A = \alpha(V_t) \subseteq D \cap V^m$ and $E = 0$, $F = D$ satisfy the lemma.