INDEPENDENT AXIOMS FOR CONVEXITY

Victor Bryant

Join-structures or Convexity Spaces generalise the geometry of Vector Spaces by means of axioms concerning line segments. Most other generalisations of this type are just particular examples of Convexity Spaces. In the many papers on this subject the collection of axioms is too long: in this short note we exhibit an independent set of axioms for these structures.

Let X be a non-empty set, let a, b, \ldots be elements of X and A, B, \ldots subsets of X. We do not distinguish between an element of X and the singleton subset which it defines. Thus in X the notation \( \in \) is redundant and is replaced by \( \subseteq \). Also we write \( A = B \) to mean A meets B or \( A \cap B \neq \emptyset \). A join is a mapping

\[ \cdot : X \times X \to 2^X, \]

i.e. it associates with each ordered pair of elements of X a subset \( a \cdot b \) (or simply \( ab \)) of X.

Given a join we can define a new operation

\[ / : X \times X \to 2^X \]

by \( a / b = \{ x : a \subseteq bx \} \). The operations \( \cdot \), \( / \) easily extend to subsets, for example \( AB \) is

\[ \bigcup \{ ab : a \subseteq A, b \subseteq B \} \].

In particular \( a(bc) \) and \( (ab)c \) are subsets of X. Note that \( A/B \cong C \) if and only if \( A = BC \).

The most common example of this situation is when
X is a real vector space and
\[ a \cdot b = \{ \lambda a + (1-\lambda) b : 0 < \lambda < 1 \}, \]
and in that case
\[ a/b = \{ \lambda a + (1-\lambda)b : \lambda > 1 \}. \]
The axioms which we consider have a strong geometric motivation based upon this example.

**DEFINITION.** A pair \((X, \cdot)\) is a Convexity Space (or Join-structure) if \(\cdot\) is a join on \(X\) satisfying

1. \(ab \not\in \emptyset\)
2. \(a/b \not\in \emptyset\)
3. \(aa = a = a/a\)
4. \(ab = ba\)
5. \((ab)c = a(bc)\)
6. \(a/b = c/d \Rightarrow ad = bc\)

for all \(a, b, c, d \in X\).

Some of the consequences of these axioms are studied in [2] and [6]. Other similar axiomatic approaches can be found, for example, in [3] - [5].

**THEOREM.** If \(\cdot\) is a join on \(X\) satisfying

1. \(a/b \not\in \emptyset\)
2. \(aa = a = a/a\)
3. \((ab)c \in a(bc)\)
4. \(a/b = c/d \Rightarrow ad = bc\)

for all \(a, b, c, d \in X\), then \((X, \cdot)\) is a Convexity Space.

**Proof.** Assume that \((X, \cdot)\) satisfies I - IV. We show that properties 1, 4 and 5 hold and it will then follow that \((X, \cdot)\) is a Convexity Space.

1. \(ab \not\in \emptyset\):

   For each \(a, b \in X\) we have, by I, \(a/b \not\in \emptyset\). Thus \(a/b = a/b\) and by IV \(ab = ba\). Hence \(ab \supset ab \cap ba \not\in \emptyset\)