A NUMERICAL STUDY OF OPTIMIZED SPARSE PRECONDITIONERS

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Abstract.

Preconditioning strategies based on incomplete factorizations and polynomial approximations are studied through extensive numerical experiments. We are concerned with the question of the optimal rate of convergence that can be achieved for these classes of preconditioners.

Our conclusion is that the well-known Modified Incomplete Cholesky factorization (MIC), cf. e.g., Gustafsson [20], and the polynomial preconditioning based on the Chebyshev polynomials, cf. Johnson, Micchelli and Paul [22], have optimal order of convergence as applied to matrix systems derived by discretization of the Poisson equation. Thus for the discrete two-dimensional Poisson equation with \( n \) unknowns, \( O(n^{\alpha}) \) and \( O(n^\gamma) \) seem to be the optimal rates of convergence for the Conjugate Gradient (CG) method using incomplete factorizations and polynomial preconditioners, respectively. The results obtained for polynomial preconditioners are in agreement with the basic theory of CG, which implies that such preconditioners can not lead to improvement of the asymptotic convergence rate.

By optimizing the preconditioners with respect to certain criteria, we observe a reduction of the number of CG iterations, but the rates of convergence remain unchanged.

AMS subject classification: 65F10, 15A06, 65F90, 65K10

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1. Introduction.

In the present paper we are concerned with efficient strategies for iterative solution of the linear system of equations

\[
(Ax = b, \quad -\Delta u(x, y) = f(x, y)
\]

where \( A \in \mathbb{R}^{n \times n} \) is symmetric and positive definite (SPD). In particular we will restrict our attention to difference approximations of the generic model problem

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defined on the unit square $\Omega = [0, 1]^2$ with suitable boundary conditions. Problems like (1.1) are commonly solved by the preconditioned conjugate gradient (PCG) method, cf. Concus et al. [13], thus directing our concern to the choice of a proper preconditioner.

When preconditioning the system (1.1), we search for a nonsingular matrix $M \in \mathbb{R}^{n \times n}$ such that the transformed system

$$M^{-1}Ax = M^{-1}b$$

can be solved in fewer PCG iterations than needed for the original problem. To do this we can choose either an implicit or explicit method. In this context the term "implicit" refers to a preconditioning matrix $M$ which in some sense approximates the original coefficient matrix $A$, such that a system of the form $Mt = r$ has to be solved in each PCG iteration. The solution of these inner systems will be a very time-consuming part of the overall process, thus suggesting that $M$ could be represented by some LU factorization such that $Mt = r$ is reduced to two triangular systems. This concept is used by preconditioning techniques based on the so called incomplete Cholesky factorizations IC, MIC and RIC, cf. papers by Meijerink and van der Vorst [25], Gustafsson [20], Axelsson and Lindskog [8]. The idea of IC is to compute the Cholesky decomposition $LL^T$ of $A$ except that $L$ is allowed to have non-zero entries only in some predefined matrix positions. Usually one chooses the sparsity pattern of $L$ equal to that of the lower triangular part of $A$. This procedure is denoted by IC(0), while IC($k$), $k > 0$, indicates that $L$ is allowed to have $k$ extra non-zero diagonals. Regarding MIC factorization, $L$ is computed as the incomplete Cholesky factor of $A + \xi h^2 I$ where $\xi > 0$ is a problem-dependent parameter, cf. Chan and Elman [12]. The RIC strategy generalizes the IC and MIC methods by introducing a relaxation parameter $\omega$, such that certain choices of $\omega$ reproduce the other two factorizations; see Chan [11]. As for the IC factorization it is possible to extend the support of the MIC or RIC factors to get MIC($k$) and RIC($k$).

An alternative to the factorized preconditioners that has received much attention in recent years, is to establish a sparse matrix $M^{-1}$ which approximates $A^{-1}$. For such preconditioners, which are referred to as being "explicit", the inner system is solved by direct computation of the product $t = M^{-1}r$. This property makes such methods very attractive for implementation on vector and/or parallel computers, whereas the implicit procedures may reduce the potential of such architectures. The explicit preconditioner $M^{-1}$ can for instance be a diagonal scaling, cf. Pini and Gambolatti [27], a truncated Neumann series for $A^{-1}$, cf. Dubois et al. [17], or a matrix-valued polynomial, cf. Johnson et al. [22].

Supposing we have the splitting $A = I - G$ where $A$ is scaled to have unit

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1 In the case of our model problem the optimal choice is $\xi \approx 2\pi^2$ for Dirichlet boundary conditions and $\xi \approx 8\pi^2$ for the periodic case. Throughout this paper we apply the proper optimal value whenever MIC is used.