ASYMPTOTIC ERROR EXPANSION FOR THE NYSTRÖM METHOD FOR NONLINEAR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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Abstract.

In this paper the asymptotic error expansion for the Nyström method for one-dimensional nonlinear Fredholm integral equations of the second kind is considered. We show that the Nyström solution admits an error expansion in powers of the step-size $h$. Thus Richardson's extrapolation can be performed on the solution, and this will greatly increase the accuracy of the numerical solution.


Key words: Richardson extrapolation, Fredholm integral equation, error expansion.

1. Introduction.

Consider the nonlinear Fredholm integral equation of the second kind

$$\int_{a}^{b} k(x, t, u(t)) \, dt + f(x), \quad x \in [a, b]$$

(1.1)

Here, $u(x)$ is an unknown function, $f(x)$ and $k(x, t, u)$ are given continuous functions defined, respectively, on $[a, b]$ and $D = ([a, b] \times [a, b] \times (-\infty, \infty))$; and without loss of generality, we assume that $k(x, t, 0) = 0$. Otherwise, since

$$\int_{a}^{b} (k(x, t, u(t)) - k(x, t, 0)) \, dt + f(x) + \int_{a}^{b} k(x, t, 0) \, dt$$

we have

(1.1')

$$u(x) = \int_{a}^{b} \hat{k}(x, t, u(t)) \, dt + \hat{f}(x)$$

where $\hat{f}(x) = f(x) + \int_{a}^{b} k(x, t, 0) \, dt$, $\hat{k}(x, t, u) = k(x, t, u) - k(x, t, 0)$, so $\hat{k}(x, t, 0) = 0$. We shall discuss (1.1').

Many methods are known for the approximate solution of (1.1) (see, for

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example [1]-[3]). But the error expansions for numerical solutions of integral equations seem to have been discussed in only a few places. There is a brief mention in [4: pp. 300–309]; and Baker [1: pp. 466–473] gives a simple analysis for the Nyström method of linear Fredholm integral equation. Under the assumption of a uniform partition, McLean [5] obtains asymptotic error expansions for numerical solutions of linear Fredholm integral equations, including the Nyström method, iterated collocation method, and iterated Galerkin method. Lin, Sloan and Xie [6] give a one-term asymptotic error expansion for the iterated collocation method of a Fredholm integral equation on an arbitrary mesh. In this paper, we consider the nonlinear Fredholm integral equation of the second kind, and we obtain an asymptotic error expansion for the Nyström solution of (1.1).

2. The Nyström method and its asymptotic expansion.

Let $\Delta$ be an equidistant partition of $[a, b]$

$$\Delta: \quad a = x_0 < x_1 < \ldots < x_N = b$$

and let $h = (b - a)/N$.

Fix $m$, and select a basis quadrature rule

$$Qf = \sum_{p=1}^{m} w_p f(t_p) \approx \int_{0}^{1} f(t) \, dt$$

whose abscissae satisfy $0 \leq t_1 < t_2 < \ldots < t_m \leq 1$, and whose weights satisfy

$$\sum_{p=1}^{m} w_p = 1$$

Thus, the rule (2.1) is exact for constant functions.

Let $x_{ij} = x_i + tjh$ ($i = 0, 1, \ldots, N - 1, j = 1, 2, \ldots, m$), from (2.1) we obtain a composite quadrature rule

$$Q_N f = \sum_{i=0}^{N-1} h \sum_{j=1}^{m} w_j f(x_{ij}) \approx \int_{a}^{b} f(t) \, dt$$

Notice that $x_i \leq x_{i1} < x_{i2} < \ldots < x_{im} \leq x_{i+1}$ for all $0 \leq i \leq N - 1$. The condition (2.2) implies that the composite rule (2.3) is convergent for every $f \in C[a, b]$.

Let $K$ be the Urysohn integral operator

$$(Ku)(x) = \int_{a}^{b} k(x, t, u(t)) \, dt$$

Define a discrete integral operator $K_N$:

$$K_N u(x) = \sum_{i=0}^{N-1} \sum_{j=1}^{m} h w_j k(x, x_{ij}, u(x_{ij}))$$