OPTIMAL STOCHASTIC QUADRATURE FORMULAS
FOR CONVEX FUNCTIONS

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Abstract.

We study optimal stochastic (or Monte Carlo) quadrature formulas for convex functions. While nonadaptive Monte Carlo methods are not better than deterministic methods, we prove that adaptive Monte Carlo methods are much better.

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1. Introduction and Result.

For each (finite) quadrature formula, the error in the class of convex functions on, e.g., [0, 1] is not uniformly bounded. For the study of optimal quadrature formulas we therefore have to restrict the class of convex functions. The classes

\[ F_{uv} = \{ f : [0, 1] \to \mathbb{R} \mid f \text{ convex}, f'_+(0) \geq u, f'_-(1) \leq v \}, \]

where \( v > u \), were studied by Glinkin (1984), Zwick (1988), and Novak (1993). The following is known for these classes.

**FACT 1.** Let \( n \in \mathbb{N} \) and \( t_i = (2i - 1)/(2n) \). Then the affine and nonadaptive formula

\[
Q_n(f) = \frac{1}{16n^2}(v - u) + \frac{1}{n} \sum_{i=1}^{n} f(t_i)
\]

is optimal even in the class of adaptive formulas of the form

\[
Q_n(f) = \phi(f(t_1), \ldots, f(t_n)),
\]

where the knot \( t_i \) may depend on the ‘already known’ function values, i.e., \( t_i = t_i(f(t_1), \ldots, f(t_{i-1})) \), and \( \phi \) is an arbitrary mapping. Defining the maximal error of \( Q_n \) on an arbitrary class \( F \) by

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We have

\[ \Delta_{\text{max}}(Q_n, F) = \sup_{f \in F} \left\{ \left| \int_0^1 f(x) \, dx - Q_n(f) \right| \right\}, \]

we have

\[ \Delta_{\text{max}}(Q_n, F_u) = \frac{v - u}{16n^2}. \]

This result has two drawbacks:

1) The optimal formula (1) depends on the class parameters \( u \) and \( v \) which might be unknown to us. It would be better to have an 'almost optimal' formula which is independent of \( u \) and \( v \). It is easy to show that the linear formula \( Q_n(f) = (1/n) \sum_{i=1}^{n} f(t_i) \) is almost optimal for all \( u \) and \( v \).

2) There are convex functions whose one-sided derivatives are not bounded, hence we would prefer larger classes without smoothness assumptions. Both these drawbacks are resolved by results of Braun (1982) who studied the classes

\[ F_1 = \{ f: [0, 1] \to \mathbb{R} \mid f \text{ convex, } \| f \|_{\infty} \leq 1 \} \]

and

\[ \tilde{F}_1 = \{ f: [0, 1] \to \mathbb{R} \mid f \text{ convex, } \max(|f(0)|, |f(1/2)|, |f(1)|) \leq 1 \} \]

and proved the following.

**FACT 2.** Let \( c_n \) be defined by

\[ c_n = \begin{cases} 2(n^2 + 2n + 1)^{-1} & \text{if } n \text{ is odd and} \\ 2(n^2 + 2n + 2)^{-1} & \text{if } n \text{ is even.} \end{cases} \]

Then the linear formula

\[ Q_n(f) = \begin{cases} c_n \left( \sum_{i=1}^{(n-1)/2} 2i f(i^2 c_n) + 2i f(1 - i^2 c_n) + nf(1/2) \right) & \text{if } n \text{ is odd and} \\ c_n \left( \sum_{i=1}^{n/2} 2i f(i^2 c_n) + 2i f(1 - i^2 c_n) \right) & \text{if } n \text{ is even} \end{cases} \]

is optimal for \( F_1 \) and also for \( \tilde{F}_1 \) in the class of all nonadaptive quadrature formulas. The maximal error of the optimal formula is given by

\[ \Delta_{\text{max}}(Q_n, F_1) = \Delta_{\text{max}}(Q_n, \tilde{F}_1) = c_n. \]

We want to make some comments concerning these results:

1) Though adaptive formulas might be slightly better than nonadaptive formulas for \( F_1 \) or \( \tilde{F}_1 \), it follows from the proofs for the classes \( F_u \) that \( n^{-2} \) is also the optimal order of convergence for adaptive formulas.

2) The class \( F_1 \) seems to be a more natural class as compared to \( \tilde{F}_1 \). It is, however, much simpler to check the membership \( f \in \tilde{F}_1 \) than to check \( f \in F_1 \).

3) \( F_1 \) (and also \( \tilde{F}_1 \)) is indeed a very general class of convex functions. If \( f \) is